MODULAR REPRESENTATIONS OF THE SUPERGROUP $Q(n)$, I

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To Professor Steinberg with admiration

1. Introduction

The representation theory of the algebraic supergroup $Q(n)$ has been studied quite intensively over the complex numbers in recent years, especially by Penkov and Serganova [18, 19, 20] culminating in their solution [21, 22] of the problem of computing the characters of all irreducible finite dimensional representations of $Q(n)$. The characters of one important family of irreducible representations, the so-called *polynomial representations*, had been determined earlier by Sergeev [24], exploiting an analogue of Schur-Weyl duality connecting polynomial representations of $Q(n)$ to the representation theory of the double covers $\widehat{S}_n$ of the symmetric groups. In [2], we used Sergeev's ideas to classify for the first time the irreducible representations of $\widehat{S}_n$ over fields of positive characteristic $p > 2$. In the present article and its sequel, we begin a systematic study of the representation theory of $Q(n)$ in positive characteristic, motivated by its close relationship to $\widehat{S}_n$.

Let us briefly summarize the main facts proved in this article by purely algebraic techniques. Let $G = Q(n)$ defined over an algebraically closed field k of characteristic $p \neq 2$, see §§2-3 for the precise definition. In §4 we construct the superalgebra $\mathrm{Dist}(G)$ of distributions on G by reduction modulo p from a Kostant $\mathbb{Z}$-form for the enveloping superalgebra of the Lie superalgebra $\mathfrak{q}(n, \mathbb{C})$. This provides one of the main tools in the remainder of the paper: there is an explicit equivalence between the category of representations of G and the category of "integrable" $\mathrm{Dist}(G)$-supermodules (see Corollary 5.7).

In §6, we classify the irreducible representations of G by highest weight theory. They turn out to be parametrized by the set

$$X_p^+(n) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \text{ with } \lambda_i = \lambda_{i+1} \text{ only if } p|\lambda_i\}.$$

For $\lambda \in X_p^+(n)$, the corresponding irreducible representation is denoted $L(\lambda)$, and is constructed naturally as the simple socle of an induced representation

$$H^0(\lambda) := \mathrm{ind}_B^G \mathfrak{u}(\lambda)$$

where B is a Borel subgroup of G and $\mathfrak{u}(\lambda)$ is a certain irreducible representation of B of dimension a power of 2. The main difficulty here is to show that $H^0(\lambda) \neq 0$ for $\lambda \in X_p^+(n)$, which we prove by exploiting the main result of [2] classifying the irreducible polynomial representations of G: so ultimately the proof that $H^0(\lambda) \neq 0$ depends on a counting argument involving p-regular conjugacy classes in $\widehat{S}_n$.

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We then turn to considering extensions between irreducible representations. Unlike for reductive algebraic groups, self-extensions are possible, arising from the fact that representations of the "diagonal" subgroup H, which plays the role of maximal torus, are not completely reducible. For example, see Lemma 7.7, there is an odd extension between the trivial module and itself. There is a "linkage principle", see Theorem 8.10, involving the notion of residue content of a weight $\lambda \in X_p^+(T)$ which already appeared in work of Leclerc and Thibon [15]. Our proof of the linkage principle is similar to the original Carter-Lusztig proof [3] of the linkage principle for $GL(n)$: there are enough explicitly known central elements in $\mathrm{Dist}(G)$ thanks to the work of Sergeev [25].

There is also an analogue of the Steinberg tensor product theorem, see Theorem 9.9. For this, we exploit the Frobenius morphism

$$F : Q(n) \to GL(n)$$

defined by raising matrix entries to the power p. Given any irreducible representation L of $GL(n)$, its Frobenius twist F^*L gives an irreducible representation of G. We show that any irreducible representation of G can be decomposed as a tensor product of such an F^*L and a *restricted* irreducible representation. Finally, the restricted irreducible representations of G are precisely the $L(\lambda)$ which remain irreducible over the Lie superalgebra of G. They are parametrized by the set

$$\begin{aligned} X_p^+(n)_{\mathrm{res}} = \{(\lambda_1, \dots, \lambda_n) \in X_p^+(n) \mid \lambda_i - \lambda_{i+1} \leq p \text{ if } p \nmid \lambda_i, \\ \lambda_i - \lambda_{i+1} < p \text{ if } p \mid \lambda_i\}. \end{aligned}$$

In particular, the determinant representation det of $GL(n)$ gives us a one dimensional representation $F^* \det$ of G of highest weight $(p, p, \dots, p)$. Thus in positive characteristic G has many one dimensional representations, unlike over $\mathbb{C}$ when there is only the trivial representation. If M is an arbitrary finite dimensional representation of G, we can tensor M with a sufficiently large power of $F^* \det$ to obtain a polynomial representation. So in positive characteristic, the category of polynomial representations is just as hard to understand as the category of all "rational" representations. To further clarify the connection between polynomial and rational representations, we show in §10 that a representation of G is polynomial if and only if it is polynomial over the diagonal subgroup H, following an argument due to Jantzen [12]. For example the representations $H^0(\lambda)$ are polynomial whenever all λ_i are non-negative.

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2. Superschemes and supergroups

Let k be a fixed algebraically closed field of characteristic $p \neq 2$. All objects considered here (vector superspaces, superalgebras, superschemes, ...) will be defined over k. By a *commutative superalgebra*, we mean a $\mathbb{Z}_2$-graded algebra $A = A_{\bar 0} \oplus A_{\bar 1}$ such that $ab = (-1)^{\bar a \bar b} ba$ for all $a, b \in A$. Here, $\bar a \in \mathbb{Z}_2$ is our notation for the degree of homogeneous $a \in A$, and the preceding formula is to be interpreted for not necessarily homogeneous a, b by extending linearly from the homogeneous case. We will write $\mathfrak{salg}_k$ for the category of all commutative superalgebras and even homomorphisms. For more of the basic notions of superalgebra, we refer to [16, ch.I] and [17, ch.3, §§1–2, ch.4, §1].

A *superscheme* $X = (X, \mathscr{O}_X)$ means a superscheme over k in the sense of [17, ch.4,§1.6]. Let $\mathfrak{ssch}_k$ denote the category of all superschemes. In this article, we will always adopt a functorial language for superschemes similar to the language of Demazure and Gabriel [5, ch.I] or Jantzen [11]. Thus, we will identify a superscheme X with its *associated functor*

$$\mathrm{Hom}_{\mathfrak{ssch}_k}(\mathrm{Spec}\,?, X) : \mathfrak{salg}_k \to \mathfrak{sets}.$$

For example, the affine superscheme $\mathbb{A}^{m|n}$ can be defined as

$$\mathbb{A}^{m|n} := \mathrm{Spec}\, k[x_1, \dots, x_m; x'_1, \dots, x'_n],$$

the latter denoting the prime spectrum of the free commutative superalgebra on even generators x_i and odd generators x'_j. With the functorial language, we are viewing $\mathbb{A}^{m|n}$ instead as the functor defined on a commutative superalgebra A by $\mathbb{A}^{m|n}(A) = A_{\bar{0}}^{\oplus m} \oplus A_{\bar{1}}^{\oplus n}$ and coordinatewise on morphisms. In general if X is an affine superscheme we write $k[X]$ for its *coordinate ring*, which we identify with the superalgebra $\mathrm{Mor}(X, \mathbb{A}^{1|1})$ of all natural transformations from the functor X to the functor $\mathbb{A}^{1|1}$. Then, using functorial language, $X = \mathrm{Hom}_{\mathfrak{salg}_k}(k[X], -)$.

By a *supergroup*, we mean a functor G from the category $\mathfrak{salg}_k$ to the category $\mathfrak{groups}$. We say that G is an *algebraic supergroup* if it is an affine superscheme (when viewed as a functor to $\mathfrak{sets}$) whose coordinate ring $k[G]$ is finitely generated as a k-superalgebra. In that case, $k[G]$ has a canonical structure of Hopf superalgebra, with comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$, antipode $S : k[G] \to k[G]$ and counit $E : k[G] \to k$ defined as the comorphisms of the multiplication, the inverse and the unit of G. The *underlying purely even group* G_{ev} is the closed subgroup of G corresponding to the Hopf superideal $k[G]k[G]_{\bar{1}}$ of the coordinate ring; as a functor, we have that $G_{\mathrm{ev}}(A) = G(A_{\bar{0}})$ for all commutative superalgebras A.

For example, for any vector superspace M, we have the supergroup $GL(M)$ with $GL(M, A)$ (for each commutative superalgebra A) equal to the group of all even A-linear automorphisms of $M \otimes A$. For finite dimensional M, $GL(M)$ is an algebraic supergroup, and the underlying even group is the algebraic group $GL(M_{\bar{0}}) \times GL(M_{\bar{1}})$.

A *representation* of an algebraic supergroup G means a natural transformation $\rho : G \to GL(M)$ for some vector superspace M. As usual, there is an equivalent module-theoretic formulation: a *G-supermodule* M means a vector superspace equipped with an even structure map $\eta_M : M \to M \otimes k[G]$ making M into a right $k[G]$-comodule in the usual sense. Given a G-supermodule M, one obtains a representation $\rho : G \to GL(M)$ by defining the action of each $G(A)$ on $M \otimes A$ by $g(m \otimes 1) = \sum_i m_i \otimes f_i(g)$, if $\eta_M(m) = \sum_i m_i \otimes f_i$. Conversely, given a representation $\rho : G \to GL(M)$, one lets $\eta_M : M \to M \otimes k[G]$ be the map $m \mapsto \rho(\mathrm{id}_{k[G]})(m \otimes 1_{k[G]})$, where $\mathrm{id}_{k[G]}$ denotes the element of $G(k[G]) \cong \mathrm{Hom}_{\mathfrak{salg}_k}(k[G], k[G])$ corresponding to the identity map.

We write $\mathfrak{mod}_G$ for the category of all G-supermodules and arbitrary (not necessarily homogeneous) homomorphisms. So the category $\mathfrak{mod}_G$ is not an abelian category. However the *underlying even category* consisting of the same objects and only even morphisms is abelian, which allows us to make use of all the usual machinery of homological algebra. We also have the *parity change functor*

$$\Pi : \mathfrak{mod}_G \to \mathfrak{mod}_G$$

defined on objects by letting ΠM equal M as a $k[G]$-comodule, but with the opposite $\mathbb{Z}_2$-grading.

In general, we write $M \simeq N$ if G-supermodules M and N are isomorphic in the underlying even category, and $M \cong N$ if they are isomorphic in $\mathfrak{mod}_G$ itself. We say that an irreducible G-supermodule M is of type $\mathtt{Q}$ if $M \simeq \Pi M$ and of type $\mathtt{M}$ otherwise. By the superalgebra analogue of Schur's lemma,

$$\mathrm{End}_G(M) \simeq \begin{cases} k & \text{if } M \text{ is of type } \mathtt{M}, \\ k \oplus \Pi k & \text{if } M \text{ is of type } \mathtt{Q}. \end{cases}$$

We have the *right regular* representation of G on $k[G]$ defined for each superalgebra A by $(gf)(h) = f(hg)$ for $f \in k[G] \otimes A = A[G_A], g \in G(A)$ and $h \in G(B)$ for all A-superalgebras B. The structure map of the associated G-supermodule is just the comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$. Instead, the *left regular* representation is defined by the formula $(gf)(h) = f(g^{-1}h)$.

3. The supergroup $Q(n)$

For the remainder of the article, G will denote the supergroup $Q(n)$. Thus, G is the functor from $\mathfrak{salg}_k$ to the category $\mathfrak{groups}$ defined on a superalgebra A by letting $G(A)$ be the group of all invertible $2n \times 2n$ matrices (under usual matrix multiplication) of the form

$$g = \left(\begin{array}{c|c} S & S' \\ \hline -S' & S \end{array} \right) \tag{3.1}$$

where S is an $n \times n$ matrix with entries in $A_{\bar{0}}$ and S' is an $n \times n$ matrix with entries in $A_{\bar{1}}$. On a morphism $f : A \to B$, $G(f) : G(A) \to G(B)$ is the group homomorphism that arises by applying f to each matrix entry. The underlying even group G_{ev} is isomorphic to $GL(n)$, being the functor mapping a superalgebra A to the group of all invertible matrices of the form (3.1) with $S' = 0$.

To see that G is an algebraic supergroup, let Mat denote the functor from $\mathfrak{salg}_k$ to $\mathfrak{sets}$ defined for a superalgebra A so that $Mat(A)$ is the set of *all* (not necessarily invertible) matrices of the form (3.1). Then Mat is isomorphic to the affine superscheme $\mathbb{A}^{n^2|n^2}$, with coordinate ring $k[Mat]$ being the free commutative superalgebra on even generators $s_{i,j}$ and odd generators $s'_{i,j}$ for $1 \leq i, j \leq n$, these being the natural transformations $Mat \to \mathbb{A}^{1|1}$ picking out the ij-entries of the matrices S and S' respectively when $g \in Mat(A)$ is represented in the form (3.1). By [16, I.7.2], a matrix $g \in Mat(A)$ is invertible if and only if $\det S \in A^{\times}$. Hence, G is the principal open subset of Mat defined by $\det : g \mapsto \det S$. In particular, the coordinate ring $k[G]$ is the localization of $k[Mat]$ at the function det.

The Hopf superalgebra structure on $k[G]$ is given explicitly in this case by the formulae

$$\Delta(s_{i,j}) = \sum_{k=1}^{n}(s_{i,k} \otimes s_{k,j} - s'_{i,k} \otimes s'_{k,j}),$$
$$\Delta(s'_{i,j}) = \sum_{k=1}^{n}(s_{i,k} \otimes s'_{k,j} + s'_{i,k} \otimes s_{k,j}),$$
$$E(s_{i,j}) = \delta_{i,j}, \quad E(s'_{i,j}) = 0$$

for $1 \leq i, j \leq n$. Note $k[Mat]$ is a subbialgebra of $k[G]$, but is not invariant under the antipode.

We have the natural G-supermodule V, namely the vector superspace $k^{n|n}$ with canonical basis $v_1, \dots, v_n, v'_1, \dots, v'_n$, where $v_1, \dots, v_n$ are even and $v'_1, \dots, v'_n$ are odd. Identify elements of $V \otimes A$ with column vectors so that

$$\sum_{i=1}^{n}(v_i \otimes a_i + v'_i \otimes a'_i) \longleftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a'_1 \\ \vdots \\ a'_n \end{pmatrix} \in A^{n|n}.$$

Then, the action of $G(A)$ on $V \otimes A$ is the obvious action on column vectors by left multiplication. In particular, the comodule structure map η satisfies

$$\eta(v_j) = \sum_{i=1}^{n}(v_i \otimes s_{i,j} - v'_i \otimes s'_{i,j}), \quad \eta(v'_j) = \sum_{i=1}^{n}(v'_i \otimes s_{i,j} + v_i \otimes s'_{i,j})$$

for each $j = 1, \dots, n$. Note V is irreducible of type $\mathtt{Q}$, since it possesses the odd automorphism

$$J : V \to V, \qquad v_i \mapsto v'_i, v'_i \mapsto -v_i. \tag{3.2}$$

Let H denote the closed subgroup of G defined on a commutative superalgebra A so that $H(A)$ consists of all matrices of the form (3.1) such that S and S' are diagonal matrices. Let B (resp. B^+) denote the subgroup with each $B(A)$ (resp. $B^+(A)$) consisting of all matrices of the form (3.1) with S and S' lower (resp. upper) triangular. The supergroup H will play the role of "maximal torus", while B is the standard Borel subgroup.

There are supergroup epimorphisms

$$\mathrm{pr} : B \to H, \qquad \mathrm{pr}^+ : B^+ \to H \tag{3.3}$$

defined for each commutative superalgebra A to be the projection of a matrix onto its diagonal part. We set $U := \ker \mathrm{pr}$ and $U^+ = \ker \mathrm{pr}^+$, respectively. Note B (resp. B^+) is the *semidirect product* of U (resp. U^+) by H, in the sense of [11, I.2.6].

We call an algebraic supergroup U *unipotent* if the fixed point space

$$\begin{aligned} M^U &:= \{m \in M \mid \eta_M(m) = m \otimes 1_{k[U]}\} \\ &= \{m \in M \mid u(m \otimes 1_A) = m \otimes 1_A \text{ for all} \\ &\qquad\qquad \text{superalgebras } A \text{ and all } u \in U(A)\} \end{aligned}$$

is non-zero for all non-zero U-supermodules M.

Lemma 3.4. *The algebraic supergroups U and U^+ are unipotent.*

Proof. One easily constructs a chain of closed normal subgroups

$$1 = U_0 < U_1 < \cdots < U_{n(n-1)} = U$$

and closed subgroups $Q_i < U$ such that for each i, U_i is the semidirect product of U_{i-1} by Q_i, with each Q_i isomorphic to one of the additive supergroups $\mathbb{A}^{1|0}$ or $\mathbb{A}^{0|1}$. Now let M be any non-zero U-supermodule; we need to show that $M^U \neq 0$. Proceeding by induction on i and using that $M^{U_i} = (M^{U_{i-1}})^{Q_i}$, it suffices to prove that the supergroups $\mathbb{A}^{1|0}$ and $\mathbb{A}^{0|1}$ are unipotent. That is well-known for $\mathbb{A}^{1|0}$, and obvious for $\mathbb{A}^{0|1}$ since its regular representation has precisely two composition factors (namely, k and Πk) both of which are trivial. □

We have the inflation functor

$$\mathrm{pr}^* : \mathfrak{mod}_H \to \mathfrak{mod}_B$$

coming from the surjection pr of (3.3); we will generally regard H-supermodules as B-supermodules by inflation in this way without further comment. Taking U-fixed points defines a right adjoint to inflation. Using this and Lemma 3.4 shows that every irreducible B-supermodule is the inflation of an irreducible H-supermodule, and similarly for B^+.

Let $T = H_{\mathrm{ev}}$. Observe that T is precisely the usual n-dimensional torus of $G_{\mathrm{ev}} \cong GL(n)$ consisting of the diagonal matrices. We denote the character group of T by $X(T)$; it is the free abelian group on generators $\varepsilon_1, \dots, \varepsilon_n$, where $\varepsilon_i : T \to \mathbb{G}_m$ picks out the ith diagonal entry. We have the root system of $GL(n)$, namely, $R = R^+ \cup -(R^+)$ where

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

We partially order $X(T)$ by the usual dominance order, so $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a sum of positive roots. The $x^\lambda := x_1^{\lambda_1} \dots x_n^{\lambda_n}$ for $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X(T)$ form a basis for the algebra $\mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm n}]$. Given any finite dimensional T-module M and $\lambda \in X(T)$, we denote by M_λ the corresponding weight space, and the *character* of M is defined as

$$\mathrm{ch}\, M := \sum_{\lambda \in X(T)} (\dim M_\lambda) x^\lambda \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm n}].$$

Let W denote the symmetric group S_n, viewed (for each commutative superalgebra A) as the subgroup of $G(A)$ consisting of matrices of the form (3.1) with S being a permutation matrix and S' being zero. The longest element of W will be denoted w_0 as usual. There is a natural left action of W on $X(T)$, hence on $\mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by permuting the x_i. Obviously, if M is a finite dimensional G-supermodule then its character is W-invariant, so lies in the algebra $\mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm n}]^W$ of symmetric functions.

Finally, we describe the "big cell" in this setting. For $1 \le m \le n$, define $\det_m$ to be the determinant of the $m \times m$ matrix $(s_{i,j})_{1 \le i,j \le m}$. We will denote the principal open subset of Mat defined by $\det_1 \det_2 \dots \det_n$ by Ω; so Ω is an affine superscheme with coordinate ring $k[\Omega]$ being the localization of $k[Mat]$ at $\det_1 \det_2 \dots \det_n$.

Theorem 3.5. *Multiplication defines an isomorphism of affine superschemes between $B \times U^+$ and Ω.*

Proof. We need to show that for every commutative superalgebra A, matrix multiplication $B(A) \times U^+(A) \to G(A)$ maps $B(A) \times U^+(A)$ bijectively onto $\Omega(A)$, namely, the set of all $g \in Mat(A)$ for which $\det_1(g) \det_2(g) \dots \det_n(g)$ is a unit in A. This can be proved by induction on n, we leave the details to the reader. □

4. Distributions

Let X be an affine superscheme and $x \in X(k)$. Then, there is a general notion of the superspace $\mathrm{Dist}(X, x)$ of *distributions* on X with support at x. The definition is an obvious extension of the purely even case, see [11, I.7]: if I_x denotes the kernel of the evaluation map $k[X] \to k, f \mapsto f(x)$, a superideal of $k[X]$, then

$$\mathrm{Dist}(X, x) := \sum_{n \ge 0} \mathrm{Dist}_n(X, x)$$

where $\mathrm{Dist}_n(X, x) \cong \left(k[X]/I_x^{n+1}\right)^*$ is the annihilator in $k[X]^*$ of I_x^{n+1}. The *tangent space at* x,

$$T_x X := (I_x / I_x^2)^*,$$

is naturally identified with a subspace of $\mathrm{Dist}(X, x)$.

Now suppose that G is an algebraic supergroup and let $e \in G(k)$ be the identity element. We make $\mathrm{Dist}(G) := \mathrm{Dist}(G, e)$ into a cocommutative Hopf superalgebra in a similar way to the purely even case [11, I.7.7], by taking the duals of the Hopf superalgebra structure maps on $k[G]$. The only significant difference is that now we identify $\mathrm{Dist}(G) \otimes \mathrm{Dist}(G) \subseteq k[G]^* \otimes k[G]^*$ with a subset of $(k[G] \otimes k[G])^*$ via the map

$$i : k[G]^* \otimes k[G]^* \hookrightarrow (k[G] \otimes k[G])^*, \quad (i(u \otimes v))(a \otimes b) = (-1)^{\bar{v}\bar{a}} u(a) v(b) \tag{4.1}$$

for $u, v \in k[G]^*, a, b \in k[G]$. The supercommutator $[.,.]$ on $\mathrm{Dist}(G)$ gives $T_e G$ the structure of a Lie superalgebra, denoted $\mathrm{Lie}(G)$. Also note that if M is a G-supermodule with structure map $\eta : M \to M \otimes k[G]$, we can view M as a left $\mathrm{Dist}(G)$-supermodule with action $um := (\mathrm{id}_M \bar{\otimes} u)\eta(m)$.

We wish to describe $\mathrm{Dist}(G)$ explicitly for $G = Q(n)$. The most concrete way to realize this is by reduction modulo p via a $\mathbb{Z}$-form of the enveloping superalgebra of the Lie superalgebra $\mathfrak{q}(n, \mathbb{C})$. So recall first that $\mathfrak{q}(n, \mathbb{C})$ is the Lie superalgebra of all matrices of the form

$$x = \left(\begin{array}{c|c} S & S' \\ \hline S' & S \end{array}\right) \tag{4.2}$$

under the supercommutator $[.,.]$, where S and S' are $n \times n$ matrices over $\mathbb{C}$, and such a matrix is even if $S' = 0$ or odd if $S = 0$. Let $\mathscr{U}_{\mathbb{C}}$ be the universal enveloping superalgebra of $\mathfrak{q}(n, \mathbb{C})$, see [13, 23]. For $1 \leq i, j \leq n$, let $e_{i,j}$ (resp. $e'_{i,j}$) denote the matrix of the form (4.2) where the ij-entry of S (resp. S') is 1, and all other entries are zero. Thus, the $\{e_{i,j}, e'_{i,j} \,|\, 1 \leq i, j \leq n\}$ form a basis of $\mathfrak{q}(n, \mathbb{C})$. By the PBW theorem for Lie superalgebras, see [23, §2.3, Cor.1], we obtain a basis for $\mathscr{U}_{\mathbb{C}}$ consisting of all monomials

$$\prod_{1 \leq i,j \leq n} e_{i,j}^{a_{i,j}} \prod_{1 \leq i,j \leq n} \left(e'_{i,j}\right)^{d_{i,j}}$$

where $a_{i,j}$ are non-negative integers, $d_{i,j} \in \{0, 1\}$, and the product is taken in any fixed order. We set $h_i := e_{i,i}, h'_i := e'_{i,i}$ for short.

Define the *Kostant $\mathbb{Z}$-form* $\mathscr{U}_{\mathbb{Z}}$ to be the $\mathbb{Z}$-subalgebra of $\mathscr{U}_{\mathbb{C}}$ generated by all elements of the form

$$e_{i,j}^{(m)},\ e'_{i,j},\ \binom{h_r}{m},\ h'_r, \quad 1 \leq i \neq j \leq n,\ 1 \leq r \leq n,\ m \geq 0,$$

where $e_{i,j}^{(m)}$ denotes the *divided power* $\frac{e_{i,j}^m}{m!}$, $\binom{h}{m}$ denotes $h(h-1)\dots(h-m+1)/(m!)$. The standard comultiplication $\delta : \mathscr{U}_{\mathbb{C}} \to \mathscr{U}_{\mathbb{C}} \otimes \mathscr{U}_{\mathbb{C}}$ and counit $\varepsilon : \mathscr{U}_{\mathbb{C}} \to \mathbb{C}$ are defined as the unique superalgebra homomorphisms with $\delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ for any $x \in \mathfrak{q}(n, \mathbb{C}) \subset \mathscr{U}_{\mathbb{C}}$, see e.g. [23, §2.4]. Also the antipode $\sigma : \mathscr{U}_{\mathbb{C}} \to \mathscr{U}_{\mathbb{C}}$ is defined as the unique superalgebra antiautomorphism with $\sigma(x) = -x$ for all $x \in \mathfrak{q}(n, \mathbb{C}) \subset \mathscr{U}_{\mathbb{C}}$. These maps restrict to give a comultiplication $\delta : \mathscr{U}_{\mathbb{Z}} \to \mathscr{U}_{\mathbb{Z}} \otimes \mathscr{U}_{\mathbb{Z}}$, an antipode $\sigma : \mathscr{U}_{\mathbb{Z}} \to \mathscr{U}_{\mathbb{Z}}$ and a counit $\varepsilon : \mathscr{U}_{\mathbb{Z}} \to \mathbb{Z}$, which make $\mathscr{U}_{\mathbb{Z}}$ into a Hopf superalgebra over $\mathbb{Z}$. Following the proof of [26, Th.2] one verifies the following:

Lemma 4.3. *The superalgebra $\mathscr{U}_{\mathbb{Z}}$ is a $\mathbb{Z}$-free $\mathbb{Z}$-module with basis given by the set of all monomials of the form*

$$\prod_{1 \leq i \neq j \leq n} e_{i,j}^{(a_{i,j})} \left(e'_{i,j}\right)^{d_{i,j}} \prod_{1 \leq i \leq n} \binom{h_i}{a_{i,i}} \left(h'_i\right)^{d_{i,i}},$$

for all $a_{i,j} \in \mathbb{Z}_{\geq 0}$ and $d_{i,j} \in \{0, 1\}$ (the product being taken in some arbitrary but fixed order).

We return to working over our fixed algebraically closed field k. Define $\mathscr{U}_k := \mathscr{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$, naturally a Hopf superalgebra over k. We will write h_i, $\binom{h_i}{m}$, $e_{i,j}^{(m)}$, h'_i, and $e'_{i,j}$ for the elements $h_i \otimes 1$, $\binom{h_i}{m} \otimes 1$, $e_{i,j}^{(m)} \otimes 1$, $h'_i \otimes 1$, and $e'_{i,j} \otimes 1$ of $\mathscr{U}_k$, respectively (in spite of potential ambiguities).

Theorem 4.4. *$\mathscr{U}_k$ is isomorphic as a Hopf superalgebra to* $\mathrm{Dist}(G)$.

Proof. In case $k = \mathbb{C}$, the isomorphism $i : \mathscr{U}_k \to \mathrm{Dist}(G)$ is induced uniquely by the Lie superalgebra isomorphism $i : \mathfrak{q}(n, \mathbb{C}) \to \mathrm{Lie}(G)$ that maps the basis $\{e_{i,j}, e'_{i,j} | 1 \leq i, j \leq n\}$ of $\mathfrak{q}(n, \mathbb{C})$ to the unique basis of $\mathrm{Lie}(G)$ dual to $\{s_{i,j}, s'_{i,j} \,|\, 1 \leq i, j \leq n\}$. For arbitrary k, the isomorphism is the reduction modulo p of the isomorphism over $\mathbb{C}$, cf. [11, II.1.12]. $\square$

We henceforth *identify* $\mathscr{U}_k$ and $\mathrm{Dist}(G)$ in this way. We can also realize each of the superalgebras $\mathrm{Dist}(H), \mathrm{Dist}(B), \mathrm{Dist}(B^+)$ etc... as subalgebras of $\mathscr{U}_k$. For instance, $\mathrm{Dist}(H)$ is the subalgebra generated by all h_i' and $\binom{h_i}{m}$, while $\mathrm{Dist}(B)$ is the subalgebra generated by $\mathrm{Dist}(H)$ together with all $e_{i,j}^{(m)}$ and $e_{i,j}'$ for $i > j$.

5. Integrable representations

For $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X(T)$ and a $\mathrm{Dist}(G)$-supermodule M, define

$$M_\lambda := \{m \in M \mid \binom{h_i}{r} m = \binom{\lambda_i}{r} m \text{ for all } i = 1, \dots, n,\ r \geq 1\}.$$

We note by [11, I.7.14] that if M is a $\mathrm{Dist}(G)$-supermodule arising from a G-supermodule in the standard way, then M_λ as defined here coincides with the λ-weight space M_λ taken with respect to the original action of T. Now call a $\mathrm{Dist}(G)$-supermodule M *integrable* if

(i) M is locally finite, i.e. it is the sum of its finite dimensional $\mathrm{Dist}(G)$-submodules;
(ii) $M = \sum_{\lambda \in X(T)} M_\lambda$.

It is easy to see that if M is a G-supermodule, then M is integrable when viewed as a $\mathrm{Dist}(G)$-supermodule. Our goal in this section is to prove the converse statement: if M is an integrable $\mathrm{Dist}(G)$-supermodule then the $\mathrm{Dist}(G)$-action lifts uniquely to make M into a G-supermodule.

Introduce the *restricted dual* $\mathrm{Dist}(G)^\diamond$ of the Hopf superalgebra $\mathrm{Dist}(G)$. By definition, this is the set of all $f \in \mathrm{Dist}(G)^*$ vanishing on some two-sided superideal I (depending on f) such the left module $\mathrm{Dist}(G)/I$ is finite dimensional and integrable. If M is any integrable $\mathrm{Dist}(G)$-supermodule with homogeneous basis $m_i (i \in I)$, we define its *coefficient space* $c(M)$ to be the subspace of $\mathrm{Dist}(G)^*$ spanned by the *coefficient functions* $f_{i,j}$ defined from

$$u m_j = (1 \bar{\otimes} u)\left(\sum_{i \in I} m_i \otimes f_{i,j}\right) \tag{5.1}$$

for all $u \in \mathrm{Dist}(G)$. Note this definition of $c(M)$ is independent of the choice of homogeneous basis. Also, if $0 \to N \to M \to Q \to 0$ is a short exact sequence of integrable $\mathrm{Dist}(G)$-supermodules, then $c(N), c(Q) \subseteq c(M)$. The following is easy to check (cf. [7, (3.1a)]):

Lemma 5.2. *$f \in \mathrm{Dist}(G)^*$ belongs to $\mathrm{Dist}(G)^\diamond$ if and only if $f \in c(M)$ for some integrable $\mathrm{Dist}(G)$-supermodule M.*

We wish to give $\mathrm{Dist}(G)^\diamond$ the structure of a Hopf superalgebra. To do this, we identify $\mathrm{Dist}(G)^* \otimes \mathrm{Dist}(G)^*$ with a subset of $(\mathrm{Dist}(G) \otimes \mathrm{Dist}(G))^*$ as in (4.1), so $f \otimes g$ corresponds to the function with $(f \otimes g)(u \otimes v) = (-1)^{\bar{g}\bar{u}} f(u) g(v)$. Then, the dual map to the comultiplication δ on $\mathrm{Dist}(G)$ gives a multiplication on $\mathrm{Dist}(G)^*$. If M and N are two integrable $\mathrm{Dist}(G)$-supermodules, then $M \otimes N$ is also integrable, and $c(M \otimes N) = c(M)c(N)$. It therefore follows from Lemma 5.2 that $\mathrm{Dist}(G)^\diamond$ is a subalgebra of $\mathrm{Dist}(G)^*$. One then checks directly from the definition of $\mathrm{Dist}(G)^\diamond$ that the dual map to the multiplication on $\mathrm{Dist}(G)$ embeds $\mathrm{Dist}(G)^\diamond$ into $\mathrm{Dist}(G)^\diamond \otimes$

$\mathrm{Dist}(G)^\diamond$, so that its restriction gives us a comultiplication on $\mathrm{Dist}(G)^\diamond$. Moreover, the restriction of σ^* gives an antipode, denoted $S : \mathrm{Dist}(G)^\diamond \to \mathrm{Dist}(G)^\diamond$, evaluation at 1 gives a counit E, and ε gives a unit.

The main point now is to identify the Hopf superalgebras $\mathrm{Dist}(G)^\diamond$ and $k[G]$. We define a map $\sim$: $k[G] \to \mathrm{Dist}(G)^*$ by setting $\tilde{f}(u) = (-1)^{\bar{f}\bar{u}}u(f)$ for all $f \in k[G], u \in \mathrm{Dist}(G)$. The $s_{i,j}$, resp. the $S(s_{i,j})$ map to the coefficient functions of the natural G-supermodule V, resp. it dual. Hence by Lemma 5.2, the image of $k[G]$ under $\sim$ is contained in $\mathrm{Dist}(G)^\diamond$. Since the Hopf superalgebra structure on $\mathrm{Dist}(G)$ is dual to that on $k[G]$ and the structure on $\mathrm{Dist}(G)^\diamond$ is dual to that on $\mathrm{Dist}(G)$, one gets at once that $\sim$ is a Hopf superalgebra homomorphism. Moreover, if $\tilde{f}(u) = 0$ for all $u \in \mathrm{Dist}(G)$, then $u(f) = 0$ for all $u \in \mathrm{Dist}_n(G)$ so that $f \in I_e^{n+1}$ for each n, i.e. $f \in \bigcap_{n\geq 0} I_e^{n+1}$. But this is zero by Krull's intersection theorem (or a direct calculation), hence $f = 0$ and $\sim$ is in fact injective.

Let us henceforward *identify* $k[G]$ with its image under $\sim$; then we wish to prove that in fact $k[G] = \mathrm{Dist}(G)^\diamond$. We need to appeal to the analogous result in the classical case of $GL(n)$. Now, $\mathrm{Dist}(G_{\mathrm{ev}})$ is the subalgebra of $\mathrm{Dist}(G)$ generated by

$$\left\{ e_{i,j}^{(m)}, \binom{h_k}{m} \,\middle|\, 1 \leq i \neq j \leq n, 1 \leq k \leq n, m \geq 0 \right\}. \tag{5.3}$$

Just as for $\mathrm{Dist}(G)$, we define integrable $\mathrm{Dist}(G_{\mathrm{ev}})$-modules, the coefficient space $c(M)$ of an integrable $\mathrm{Dist}(G_{\mathrm{ev}})$-module M, the restricted dual $\mathrm{Dist}(G_{\mathrm{ev}})^\diamond$ and so on. Note that the restriction of an integrable $\mathrm{Dist}(G)$-supermodule to $\mathrm{Dist}(G_{\mathrm{ev}})$ is integrable as a $\mathrm{Dist}(G_{\mathrm{ev}})$-module by definition, so restriction of functions gives us a natural Hopf superalgebra homomorphism

$$\xi : \mathrm{Dist}(G)^\diamond \to \mathrm{Dist}(G_{\mathrm{ev}})^\diamond.$$

It is well-known that $\mathrm{Dist}(G_{\mathrm{ev}})^\diamond$ is *equal* to the coordinate ring $k[G_{\mathrm{ev}}]$, i.e. it is the localization of the free polynomial algebra $k[c_{i,j} \mid 1 \leq i, j \leq n]$ where $c_{i,j} := \xi(s_{i,j})$ at determinant. In other words, as goes back at least to Kostant [14] (or see [7, (3.1c)]), we have that:

Lemma 5.4. $\mathrm{Dist}(G_{\mathrm{ev}})^\diamond$ *is generated as an algebra by* $\{c_{i,j}, S(c_{i,j}) \mid 1 \leq i, j \leq n\}$.

Fix for the remainder of the section some order for products in the PBW monomials so that every ordered PBW monomial is of the form mu with m being a product of the $e'_{i,j}$ and $u \in \mathrm{Dist}(G_{\mathrm{ev}})$. Let Υ denote the resulting PBW basis of $\mathrm{Dist}(G)$, see Lemma 4.3. Set $\Gamma = \{(i,j) \mid 1 \leq i, j \leq n\}$ and for $I \subseteq \Gamma$, let $m_I \in \Upsilon$ denote the ordered PBW monomial $\prod_{(i,j)\in I} e'_{i,j}$. By Lemma 4.3, we have a direct sum decomposition

$$\mathrm{Dist}(G) = \bigoplus_{I\subseteq\Gamma} m_I \mathrm{Dist}(G_{\mathrm{ev}}),$$

showing that $\mathrm{Dist}(G)$ is a free right $\mathrm{Dist}(G_{\mathrm{ev}})$-supermodule with basis $\{m_I \mid I \subseteq \Gamma\}$. For $I \subseteq \Gamma$, define $\eta_I \in \mathrm{Dist}(G)^*$ to be the "indicator function" of the monomial m_I, i.e. $\eta_I(m_I) = 1$ and $\eta_I(m) = 0$ for any other ordered PBW monomial $m \in \Upsilon$ different from m_I.

Lemma 5.5. *For each* $I \subseteq \Gamma$, $\eta_I \in k[G]$.

Proof. Let $N = n^2$, and set $M = \bigwedge^N (V \otimes V^*)$, itself naturally a $\mathrm{Dist}(G)$-supermodule. Since both $c(V)$ and $c(V^*)$ lie in $k[G]$ by definition, we certainly have that $c((V \otimes V^*)^{\otimes N}) \subseteq k[G]$. But M is a quotient of $(V \otimes V^*)^{\otimes N}$, so $c(M) \subseteq k[G]$ too.

Recall that $v_1, \dots, v_n, v'_1, \dots, v'_n$ denotes the natural basis of V. Set $v_{n+i} := v'_i$ for short, and let $w_1, \dots, w_{2n}$ denote the basis for V^* dual to $v_1, \dots, v_{2n}$. Set $z_{i,j} := v_i \otimes w_j \in V \otimes V^*$ for $i, j = 1, \dots, 2n$. Fix also some linear order on the set of all pairs $\{(i,j) \,|\, i,j = 1, \dots, 2n\}$. Denote by Σ the set of all weakly increasing sequences $S = ((i_1, j_1) \leq (i_2, j_2) \leq \dots \leq (i_N, j_N))$ of length N such that $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if z_{i_s, j_s} is even. If $S \in \Sigma$, we denote by z_S the canonical image of the element $z_{i_1,j_1} \otimes \dots \otimes z_{i_N,j_N} \in (V \otimes V^*)^{\otimes N}$ in M. Then $\mathscr{B} := \{z_S \mid S \in \Sigma\}$ is a homogeneous basis for M. Moreover, letting $z := z_S$ where $S \in \Sigma$ is the sequence which contains every $(i,j) \in \Gamma$, z spans the trivial $\mathrm{Dist}(G_{\mathrm{ev}})$-submodule $\bigwedge^N (V_{\bar{0}} \otimes V_{\bar{0}}^*)$ of M.

The crucial step at this point is to observe that the vectors $\{m_I z \mid I \subseteq \Gamma\}$ are linearly independent. This follows because they are related to the basis $\mathscr{B}$ in a unitriangular way. Let $\mathscr{C}$ be some homogeneous basis for M extending $\{m_I z | I \subseteq \Gamma\}$. For $I \subseteq \Gamma$ and $u \in \mathrm{Dist}(G)$, define $g_I(u)$ to be the $m_I z$-coefficient of uz when expressed in terms of the basis $\mathscr{C}$. So, $g_I \in k[G]$ by the first paragraph. It is clear from the definition that $g_I(m_J) = \delta_{I,J}$ for every $I, J \subseteq \Gamma$. Moreover, since z spans a trivial $\mathrm{Dist}(G_{\mathrm{ev}})$-module, $uz = 0$ for all ordered PBW monomials $u \in \Upsilon$ not of the form m_J for any $J \subseteq \Gamma$. Hence $g_I(u) = 0$ for such u, and we have now checked that $g_I = \eta_I$. □

Now we obtain the main result:

Theorem 5.6. $\mathrm{Dist}(G)^\diamond = k[G]$.

Proof. We first claim that $(\eta_I f)(m_I u) = f(u)$ and $(\eta_I f)(m_J u) = 0$ for any $u \in \mathrm{Dist}(G_{\mathrm{ev}})$, $J \not\supseteq I$, and $f \in \mathrm{Dist}(G)^\diamond$. Indeed, by definition of the product in $\mathrm{Dist}(G)^\diamond$, for any $J \subseteq \Gamma$ we have $(\eta_I f)(m_J u) = (\eta_I \bar{\otimes} f)(\delta(m_J u))$. But, when expanded in terms of the PBW basis of $\mathrm{Dist}(G) \otimes \mathrm{Dist}(G)$, the $(m_I \otimes ?)$-component of $\delta(m_J u)$ is equal $m_I \otimes u$ if $J = I$ and 0 if $J \not\supseteq I$. This implies the claim.

Now let $f \in \mathrm{Dist}(G)^\diamond$ and $\Delta(f) = \sum_j f_j \otimes g_j$. The restrictions $\xi(g_j)$ belong to $\mathrm{Dist}(G_{\mathrm{ev}})^\diamond$, so by Lemma 5.4, there exist degree $\bar{0}$ elements $h_j \in k[G]$ with $\xi(g_j) = \xi(h_j)$. For any $I \subseteq \Gamma$, define $f_I := \sum_j f_j(m_I) \eta_I h_j$, which is an element of $k[G]$ thanks to 5.5. By the previous paragraph, we have

$$\begin{aligned} f_I(m_I u) &= \sum_j f_j(m_I)(\eta_I h_j)(m_I u) = \sum_j f_j(m_I) h_j(u) \\ &= \sum_j f_j(m_I) g_j(u) = f(m_I u). \end{aligned}$$

Similarly $f_I(m_J u) = 0$ for any $u \in \mathrm{Dist}(G_{\mathrm{ev}})$ and any $J \not\supseteq I$. Thus, we have proved that given $f \in \mathrm{Dist}(G)^\diamond$ one can find a function $f_I \in k[G]$, with $f_I = f$ on $m_I \mathrm{Dist}(G_{\mathrm{ev}})$ and $f_I = 0$ on $\bigoplus_{J \not\supseteq I} m_J \mathrm{Dist}(G_{\mathrm{ev}})$.

Now we can prove that $f \in k[G]$. For $i = 0, 1, \dots, n^2$, define the functions $f^{(i)} \in k[G]$ by setting $f^{(0)} := f - f_\varnothing$, and

$$f^{(i)} = f^{(i-1)} - \sum_{I \in \Gamma, |I| = i} (f^{(i-1)})_I.$$

By induction on i we prove that $f^{(i)} = 0$ on $\bigoplus_{J, |J| \leq i} m_J \mathrm{Dist}(G_{\mathrm{ev}})$. In particular, $f^{(n^2)}$ is the zero element of $\mathrm{Dist}(G)^\diamond$. It remains to note that $f^{(n^2)}$ is obtained from f by subtracting elements of $k[G]$. □

The theorem has the following important consequence:

Corollary 5.7. *The category $\mathfrak{mod}_G$ is isomorphic to the category of all integrable* $\mathrm{Dist}(G)$*-supermodules.*

Proof. Suppose M is an integrable $\mathrm{Dist}(G)$-supermodule with homogeneous basis $m_i (i \in I)$. Define the coefficient functions $f_{i,j} \in \mathrm{Dist}(G)^\diamond$ as in (5.1). Let $g_{i,j}$ be the element of $k[G]$ corresponding to $f_{i,j}$ under the identification, i.e. $\tilde{g}_{i,j} = f_{i,j}$. Then, we define a structure map η making M into a G-supermodule by setting

$$\eta(m_j) = \sum_i m_i \otimes g_{i,j}.$$

Conversely, if M is a G-supermodule, we view M as a $\mathrm{Dist}(G)$-supermodule in the standard way. Now check that the two constructions are inverse to one another and that morphisms of G-supermodules correspond to morphisms of $\mathrm{Dist}(G)$-supermodules. □

Because of this corollary, we will not distinguish between G-supermodules and integrable $\mathrm{Dist}(G)$-supermodules in the sequel.

6. Highest weight theory

Let H be any closed subgroup of an algebraic supergroup G. There are restriction and induction functors

$$\mathrm{res}^G_H : \mathfrak{mod}_G \to \mathfrak{mod}_H, \qquad \mathrm{ind}^G_H : \mathfrak{mod}_H \to \mathfrak{mod}_G.$$

The induction here is essentially "coalgebra induction" from [6, §3], and is defined as follows. If M is any vector superspace, we can view $M \otimes k[G]$ as a G-supermodule with structure map $\mathrm{id}_M \otimes \Delta_G$; we will denote this by $M_{tr} \otimes k[G]$ to emphasize that the G-supermodule structure is trivial on M. Let $\delta : k[G] \to k[H] \otimes k[G]$ be the comorphism of the multiplication $\bar{\mu} : H \times G \to G$. Then $\mathrm{ind}^G_H M$ is defined to be the kernel of the map $\partial = \eta \otimes \mathrm{id}_{k[G]} - \mathrm{id}_M \otimes \delta$ in the following exact sequence of G-supermodules:

$$0 \longrightarrow \mathrm{ind}^G_H M \longrightarrow M_{tr} \otimes k[G] \xrightarrow{\partial} M_{tr} \otimes k[H]_{tr} \otimes k[G]. \tag{6.1}$$

On a morphism $f : M \to M'$ of H-supermodules, $\mathrm{ind}^G_H f$ is simply the restriction of the map $f \otimes \mathrm{id}_{k[G]}$ to the subspace $\mathrm{ind}^G_H M \subseteq M_{tr} \otimes k[G]$. We remark that there is an equivalent way of characterizing $\mathrm{ind}^G_H M$ as a subspace of $M_{tr} \otimes k[G]$:

$$\mathrm{ind}^G_H M = (M \otimes k[G])^H \tag{6.2}$$

where the H-fixed points are taken with respect to the given action on M and the left regular action on $k[G]$.

Induction and restriction have the familiar properties (see e.g. [6, p.306]): for example, for a G-supermodule M, its structure map η determines a natural even isomorphism $M \xrightarrow{\sim} \mathrm{ind}_G^G M$. The functor res_H^G is exact and ind_H^G is right adjoint to res_H^G, hence is left exact and sends injectives to injectives. Now one can prove that $\mathfrak{mod}_G$ has enough injectives, by following the usual proof in the even case [11, I.3.9]. It therefore makes sense to consider the right derived functors $R^i\mathrm{ind}_H^G$: $\mathfrak{mod}_H \to \mathfrak{mod}_G$. We have the generalized tensor identity: for any G-supermodule M and H-supermodule N, there is a natural isomorphism

$$R^i\mathrm{ind}_H^G(\mathrm{res}_H^G M \otimes N) \simeq M \otimes R^i\mathrm{ind}_H^G N \tag{6.3}$$

of G-supermodules. (We wrote down a proof based on the arguments of [9, 1.3] and [11, I.4.8].)

Turn now to our case $G = Q(n)$. We wish to use induction from the Borel subgroup B of G to classify the irreducible G-supermodules by their highest weights. We should start with the case $Q(1)$. To understand this case, we analyze its coordinate ring. We know that $k[Q(1)] = k[s, s^{-1}, s']$ (where $s = s_{1,1}, s' = s'_{1,1}$), so it has basis

$$\{s^m, s^{m-1}s' \mid m \in \mathbb{Z}\}.$$

Let $k[Q(1)]_m$ be the subspace spanned by $s^m, s^{m-1}s'$, making $k[Q(1)]$ into a $\mathbb{Z}$-graded superalgebra. The comultiplication satisfies

$$\begin{aligned} \Delta(s^m) &= s^m \otimes s^m - ms^{m-1}s' \otimes s^{m-1}s', \\ \Delta(s^{m-1}s') &= s^{m-1}s' \otimes s^m + s^m \otimes s^{m-1}s', \end{aligned}$$

showing that each $k[Q(1)]_m$ is a two dimensional subcoalgebra of $k[Q(1)]$. The dual superalgebra $k[Q(1)]_m^*$ is generated by the odd element c_m defined from $c_m(s^{m-1}s') = 1, c_m(s^m) = 0$, subject only to the relation $c_m^2 = m$. So it is either a rank one Clifford superalgebra (if m is non-zero modulo p) or a rank one Grassmann superalgebra (if m is zero modulo p). In either case, $k[Q(1)]_m^*$ has a unique irreducible supermodule $\mathfrak{u}(m)$ up to isomorphism. Now one proceeds in exactly the same way as in [2, §6] to conclude that the corresponding $Q(1)$-supermodules $\{\mathfrak{u}(m) | m \in \mathbb{Z}\}$ form a complete set of pairwise non-isomorphic irreducible $Q(1)$-supermodules, with $\mathfrak{u}(m)$ being of type $\mathtt{M}$ with character x_1^m if $p|m$ and of type $\mathtt{Q}$ with character $2x_1^m$ if $p \nmid m$.

Notice that $H \cong Q(1) \times \cdots \times Q(1)$ (n times). So using the previous paragraph and the general theory of representations of direct products of supergroups (which is entirely similar to [2, Lemma 2.9]), we obtain the following parametrization of the irreducible H-supermodules. Given $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$, we define $h_{p'}(\lambda)$ to be the number of $i = 1, \dots, n$ for which $p \nmid \lambda_i$.

Lemma 6.4. *For each $\lambda \in X(T)$, there is a unique irreducible H-supermodule $\mathfrak{u}(\lambda)$ (up to isomorphism) with character $2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor} x^\lambda$. The $\{\mathfrak{u}(\lambda) \mid \lambda \in X(T)\}$ form a complete set of pairwise non-isomorphic irreducible H-supermodules. Moreover, $\mathfrak{u}(\lambda)$ is of type $\mathtt{M}$ if $h_{p'}(\lambda)$ is even, type $\mathtt{Q}$ if $h_{p'}(\lambda)$ is odd.*

As explained in §3, it follows immediately that the (inflations of) $\{\mathfrak{u}(\lambda) | \lambda \in X(T)\}$ give a complete set of pairwise non-isomorphic irreducible B-supermodules. Now we define

$$H^0(\lambda) := \mathrm{ind}_B^G \mathfrak{u}(\lambda), \qquad L(\lambda) := \mathrm{soc}_G H^0(\lambda), \tag{6.5}$$

for each $\lambda \in X(T)$. Let

$$X^+(T) := \{\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X(T) \mid \lambda_1 \geq \cdots \geq \lambda_n\}, \tag{6.6}$$

$$X_p^+(T) := \{\lambda \in X^+(T) \mid \lambda_i = \lambda_{i+1} \text{ for some } 1 \leq i < n \text{ implies } p|\lambda_i\}. \tag{6.7}$$

We refer to weights λ lying in $X^+(T)$ as *dominant weights*, and elements of $X_p^+(T)$ are *p-strict dominant weights*. We will show that the $\{L(\lambda) \mid \lambda \in X_p^+(T)\}$ form a complete set of pairwise non-isomorphic irreducible G-supermodules.

Lemma 6.8. *Let $\xi : k[G] \to k[B]$ (resp. $\xi^+ : k[G] \to k[B^+]$) denote the comorphism of the inclusion $B \to G$ (resp. the inclusion $B^+ \to G$). Then, for any H-supermodule M, the restriction of the map $\mathrm{id}_M \otimes \xi^+ : M_{tr} \otimes k[G] \to M_{tr} \otimes k[B^+]$ to $\mathrm{ind}_B^G M \subset M \otimes k[G]$ defines a monomorphism*

$$\mathrm{ind}_B^G M \hookrightarrow \mathrm{ind}_H^{B^+} M$$

as B^+-supermodules.

Proof. To prove injectivity, let $\eta : M \to M \otimes k[B]$ be the structure map of M as a B-supermodule. Take $v \in \mathrm{ind}_B^G M$ with $(\mathrm{id}_M \otimes \xi^+)(v) = 0$. By definition of induction, we have that

$$\eta \otimes \mathrm{id}_{k[G]}(v) = \mathrm{id}_M \otimes([\xi \otimes \mathrm{id}_{k[G]}] \circ \Delta_G)(v).$$

So applying $\mathrm{id}_M \otimes \mathrm{id}_{k[B]} \otimes \xi^+$ to both sides, we get that

$$0 = (\eta \otimes \mathrm{id}_{k[B^+]}) \circ (\mathrm{id}_M \otimes \xi^+)(v) = \mathrm{id}_M \otimes([\xi \otimes \xi^+] \circ \Delta_G)(v).$$

Now Theorem 3.5 implies that the map

$$(\xi \otimes \xi^+) \circ \Delta_G : k[G] \to k[B] \otimes k[B^+]$$

is injective. Hence, $v = 0$ as required. Finally, the fact that the image lies in $\mathrm{ind}_H^{B^+} M \subseteq M \otimes k[B^+]$ follows from (6.2). $\square$

Lemma 6.9. *View $k[U^+]$ as a B^+-supermodule with structure map being the comorphism ρ^* of the right action $\rho : U^+ \times B^+ \to U^+, (u, b) \mapsto \mathrm{pr}^+(b)^{-1} ub$, where $\mathrm{pr}^+ : B^+ \to H$ is the projection (3.3). Then, for any H-supermodule M,*

$$\mathrm{ind}_H^{B^+} M \simeq M \otimes k[U^+]$$

as a B^+-supermodule.

Proof. By definition, $\mathrm{ind}_H^{B^+} M$ is a B^+-submodule of $M_{tr} \otimes k[B^+]$. View $k[H] \otimes k[U^+]$ as a B^+-supermodule via the inflation of the right regular H-action on $k[H]$ and as described above on $k[U^+]$. Let $\mu : H \times U^+ \to B^+$ be the superscheme isomorphism induced by multiplication. Then the map

$$\mathrm{id}_M \otimes \mu^* : M_{tr} \otimes k[B^+] \to M_{tr} \otimes k[H] \otimes k[U^+]$$

is an isomorphism of B^+-supermodules. By definition of induction, it maps $\mathrm{ind}_H^{B^+} M$ to the subspace $\eta(M) \otimes k[U^+]$ of $M \otimes k[H] \otimes k[U^+]$, where $\eta : M \to M \otimes k[H]$ is the structure map of M. Finally, we observe that $\eta(M) \simeq M$ as B^+-supermodules, and the lemma follows. $\square$

Lemma 6.10. *Suppose that $\lambda \in X(T)$ such that $H^0(\lambda) \neq 0$.*
(i) *If μ is a weight with $H^0(\lambda)_\mu \neq 0$, then $w_0\lambda \leq \mu \leq \lambda$ in the dominance order.*
(ii) *The B^+-socle of $H^0(\lambda)$ is precisely its λ-weight space $H^0(\lambda)_\lambda \simeq \mathfrak{u}(\lambda)$.*
(iii) *$H^0(\lambda)$ is finite dimensional.*

Proof. By Lemmas 6.8 and 6.9, there is an injective B^+-supermodule map

$$H^0(\lambda) \hookrightarrow \mathrm{ind}_H^{B^+} \mathfrak{u}(\lambda) \simeq \mathfrak{u}(\lambda) \otimes k[U^+]$$

which is non-zero by the assumption on λ. By Frobenius reciprocity, $\mathrm{ind}_H^{B^+} \mathfrak{u}(\lambda)$ has B^+-socle $\simeq \mathfrak{u}(\lambda)$, hence $H^0(\lambda)$ has B^+-socle $\simeq \mathfrak{u}(\lambda)$ too. Considering the weights of $\mathfrak{u}(\lambda) \otimes k[U^+]$ gives that $\dim H^0(\lambda)_\lambda \leq \dim \mathfrak{u}(\lambda)$ and that $H^0(\lambda)_\mu = 0$ unless $\mu \leq \lambda$. Parts (i) and (ii) follow easily. Finally, for (iii), each weight space of $\mathfrak{u}(\lambda) \otimes k[U^+]$ is finite dimensional, hence each weight space of $H^0(\lambda)$ is also finite dimensional. But by (i) there are only finitely many non-zero weight spaces, hence $H^0(\lambda)$ itself must be finite dimensional. $\square$

Now we are ready to prove the main result. Note in case $p = 0$, the following theorem is due to Penkov [18].

Theorem 6.11. *For $\lambda \in X(T)$, $H^0(\lambda)$ is non-zero if and only if $\lambda \in X_p^+(T)$. The modules $\{L(\lambda) \mid \lambda \in X_p^+(T)\}$ form a complete set of pairwise non-isomorphic irreducible G-supermodules. Moreover $L(\lambda)$ is of type* `M` *if $h_{p'}(\lambda)$ is even, type* `Q` *if $h_{p'}(\lambda)$ is odd.*

Proof. Note it is immediate from Lemma 6.10(ii) that the non-zero $L(\lambda)$ are irreducible and pairwise non-isomorphic. Moreover, a Frobenius reciprocity argument shows that every irreducible G-supermodule is isomorphic to the socle of some non-zero $H^0(\lambda)$. To check the statement about the type of $L(\lambda)$ (whenever it is non-zero), note that the B-head of $L(\lambda)$ is $\cong \mathfrak{u}(\lambda)$; this follows on conjugating with w_0 and taking duals from the fact that the B^+-socle of $L(-w_0\lambda)$ is $\mathfrak{u}(-w_0\lambda)$. So we can calculate using Frobenius reciprocity:

$$\mathrm{End}_G(L(\lambda)) \cong \mathrm{Hom}_G(L(\lambda), H^0(\lambda)) \cong \mathrm{Hom}_B(L(\lambda), \mathfrak{u}(\lambda)) \cong \mathrm{End}_B(\mathfrak{u}(\lambda)).$$

Hence, $L(\lambda)$ has the same type as $\mathfrak{u}(\lambda)$ whenever it is non-zero, see Lemma 6.4.

It now just remains to prove that

$$X_p^+(T) = \{\lambda \in X(T) \mid H^0(\lambda) \neq 0\}.$$

Suppose first that $\lambda \in X(T)$ and $H^0(\lambda) \neq 0$. Since the set of weights of $H^0(\lambda)$ is invariant under the action of W, Lemma 6.10(i) implies that $\lambda_1 \geq \cdots \geq \lambda_n$. Now suppose that $\lambda_i = \lambda_{i+1}$ for some i; we need to show that $p|\lambda_i$. By restricting to the subgroup of G isomorphic to $Q(2)$ corresponding to the ith and $(i+1)$th matrix rows and columns, it suffices to do this in the case $n = 2$, so $\lambda = (\lambda_1, \lambda_2) \in X(T)$ and $\lambda_1 = \lambda_2$. Then, by Lemma 6.10(i), λ is the only non-zero weight of $H^0(\lambda)$. So if we take $0 \neq v \in H^0(\lambda)_\lambda$ and consider the action of $\mathrm{Dist}(G)$, we must have that $e'_{2,1}v = 0$. Hence,

$$e'_{1,2}e'_{2,1}v = (h_1 + h_2)v = 2\lambda_1 v = 0,$$

so $p|\lambda_1$ as required.

Conversely take $\lambda \in X_p^+(T)$. We need to show that there exists an irreducible G-supermodule L with $L_\lambda \neq 0$ and $L_\mu = 0$ for all $\mu \not\leq \lambda$. Write λ as $\lambda_- + \lambda_+$ where λ_+ (resp. λ_-) is obtained from λ by replacing all negative (resp. positive) parts by zero. By [2, Theorem 10.1], there exist irreducible G-supermodules $L(\lambda_+)$ and $L(-w_0\lambda_-)$ with highest weights λ_+ and $-w_0\lambda_-$ respectively. Then, $M := L(\lambda_+) \otimes L(-w_0\lambda_-)^*$ is a G-supermodule with $M_\lambda \neq 0$ and $M_\mu = 0$ for $\mu \not\leq \lambda$. So at least one of the composition factors of M must be an irreducible G-supermodule with highest weight λ. □

Let us finally introduce the G-supermodules

$$V(\lambda) := H^0(-w_0\lambda)^* \tag{6.12}$$

for each $\lambda \in X_p^+(T)$. Note $V(\lambda)$ is generated by the B^+-stable submodule $V(\lambda)_\lambda \cong \mathfrak{u}(\lambda)$. The $V(\lambda)$ are the "universal" highest weight modules according to the following lemma, proved as in [11, II.2.13]:

Lemma 6.13. *Let $\lambda \in X_p^+(T)$. If M is a G-supermodule which is generated by a B^+-stable submodule isomorphic to $\mathfrak{u}(\lambda)$, then M is isomorphic to a quotient of $V(\lambda)$.*

7. Extensions

Next, we consider another basic principle of highest weight theory: it should be possible to compute extensions between $L(\lambda)$ and other irreducibles by looking at $\mathrm{soc}_G(H^0(\lambda)/L(\lambda))$, cf. [11, II.2.14].

Let $\widetilde{\mathfrak{u}}(\lambda)$ denote the injective hull of $\mathfrak{u}(\lambda)$ as an H-supermodule. By reducing to the case $H = Q(1)$, one checks:

$$\dim \widetilde{\mathfrak{u}}(\lambda) = 2^{n-h_{p'}(\lambda)} \dim \mathfrak{u}(\lambda). \tag{7.1}$$

For $\lambda \in X_p^+(T)$, define

$$\widetilde{H}^0(\lambda) := \mathrm{ind}_B^G \widetilde{\mathfrak{u}}(\lambda). \tag{7.2}$$

Since ind_B^G is left exact, there is a canonical embedding $H^0(\lambda) \hookrightarrow \widetilde{H}^0(\lambda)$. Moreover, arguing using Lemma 6.8 as we did in the proof of Lemma 6.10,

$$\mathrm{soc}_{B^+}\widetilde{H}^0(\lambda) = \mathrm{soc}_{B^+}H^0(\lambda) \simeq \mathfrak{u}(\lambda), \tag{7.3}$$

$$\mathrm{soc}_G\widetilde{H}^0(\lambda) = \mathrm{soc}_G H^0(\lambda) \simeq L(\lambda). \tag{7.4}$$

Theorem 7.5. *Let $\lambda, \mu \in X_p^+(T)$ with $\mu \not\geq \lambda$.*

(i) $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \simeq \mathrm{Hom}_G(L(\mu), \widetilde{H}^0(\lambda)/L(\lambda)) \simeq \mathrm{Hom}_G(L(\mu), H^0(\lambda)/L(\lambda))$.

(ii) $\mathrm{Ext}^1_G(L(\lambda), L(\lambda)) \simeq \mathrm{Hom}_G(L(\lambda), \widetilde{H}^0(\lambda)/L(\lambda))$.

Proof. Take *arbitrary* $\lambda, \mu \in X_p^+(T)$. We have the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_G(L(\mu), L(\lambda)) \xrightarrow{\sim} \mathrm{Hom}_G(L(\mu), \widetilde{H}^0(\lambda)) \\ &\longrightarrow \mathrm{Hom}_G(L(\mu), \widetilde{H}^0(\lambda)/L(\lambda)) \longrightarrow \mathrm{Ext}^1_G(L(\mu), L(\lambda)) \\ &\longrightarrow \mathrm{Ext}^1_G(L(\mu), \widetilde{H}^0(\lambda)) \longrightarrow \dots . \end{aligned}$$

So it suffices for the first isomorphism in (i) and (ii) to show that $\mathrm{Ext}^1_G(L(\mu), \widetilde{H}^0(\lambda)) = 0$, providing $\mu \not> \lambda$. So suppose that we have an extension

$$0 \longrightarrow \widetilde{H}^0(\lambda) \xrightarrow{f} M \longrightarrow L(\mu) \longrightarrow 0.$$

Let $i : \widetilde{H}^0(\lambda) \to \widetilde{\mathfrak{u}}(\lambda)$ be the B-supermodule homomorphism induced by the identity map $\widetilde{H}^0(\lambda) \to \mathrm{ind}_B^G \widetilde{\mathfrak{u}}(\lambda)$ under adjointness. By injectivity of $\widetilde{\mathfrak{u}}(\lambda)$ and the assumption $\mu \not> \lambda$, we can find a B-supermodule homomorphism $g : M \to \widetilde{\mathfrak{u}}(\lambda)$ such that $g \circ f = i$. This induces a G-supermodule homomorphism $\bar{g} : M \to \widetilde{H}^0(\lambda)$ which splits f. This proves the claim.

The second isomorphism in (i) is proved similarly but using the fact that

$$\mathrm{Ext}^1_H(\mathfrak{u}(\lambda), \mathfrak{u}(\mu)) = 0$$

for $\lambda \neq \mu$, instead of injectivity of $\widetilde{\mathfrak{u}}(\lambda)$. □

As an application of the theorem, we compute $\mathrm{Ext}^1_G(k, k)$. Note $\mathfrak{u}(0) = k$. By (7.1), $\widetilde{\mathfrak{u}}(0)$ has dimension 2^n. Set

$$\widetilde{k} := \mathrm{ind}_B^G \widetilde{\mathfrak{u}}(0). \tag{7.6}$$

Lemma 7.7. *There is a non-split short exact sequence*

$$0 \longrightarrow k \longrightarrow \widetilde{k} \longrightarrow \Pi k \longrightarrow 0$$

of G-supermodules.

Proof. Let $M = \mathrm{Dist}(H) \otimes_{\mathrm{Dist}(T)} (\Pi^n k)$. This is an integrable $\mathrm{Dist}(H)$-supermodule, hence an H-supermodule. Note M has basis

$$h_A := h'_{a_1} \dots h'_{a_n} \otimes 1$$

indexed by subsets $A = \{a_1 < \dots < a_n\}$ of $\{1, \dots, n\}$. Let M_d denote the subspace of M spanned by all such basis elements with $|A| = d$, so $h'_i M_d \subseteq M_{d+1}$ for each i. One easily shows using this that $\mathrm{soc}_H M = M_n \simeq k$. Hence, $M \simeq \widetilde{\mathfrak{u}}(0)$.

Now $\mathrm{ind}_B^G k \simeq k$ and $\widetilde{\mathfrak{u}}(0)$ has a composition series with all composition factors $\cong k$. Hence, all composition factors of $\mathrm{ind}_B^G \widetilde{\mathfrak{u}}(0)$ are $\cong k$ too. This shows that the natural map $\mathrm{ind}_B^G \widetilde{\mathfrak{u}}(0) \hookrightarrow \widetilde{\mathfrak{u}}(0) \otimes k[U^+]$ arising from Lemmas 6.8 and 6.9 defines an embedding $\widetilde{k} = \mathrm{ind}_B^G \widetilde{\mathfrak{u}}(0) \hookrightarrow \widetilde{\mathfrak{u}}(0)$. This identifies $\widetilde{k}$ with the largest submodule of $\widetilde{\mathfrak{u}}(0)$ for which the action of H can be extended (necessarily uniquely) to an action of G.

So now suppose that N is a submodule of M maximal subject to the condition that the action of $\mathrm{Dist}(H)$ extends to an action of $\mathrm{Dist}(G)$. Clearly each h_i and each $e_{i,j}, e'_{i,j}$ for $i \neq j$ must act as zero, hence $[e_{i,j}, e'_{i,j}] = h'_i - h'_j$ acts as zero. This shows that $N = \{m \in M \mid h'_1 m = \dots = \bar{h}'_n m\}$. Now a straightforward induction gives that N has dimension 2 on basis

$$h'_1 \dots h'_n \otimes 1, \qquad \sum_{i=1}^n (-1)^i h'_1 \dots h'_{i-1} h'_{i+1} \dots h'_n \otimes 1.$$

Since $N \simeq \widetilde{k}$, this proves the lemma. □

Corollary 7.8. $\mathrm{Ext}^1_G(k, k) \simeq \Pi k$.

8. The linkage principle

In [25], Sergeev constructed certain central elements of $\mathrm{Dist}(G)$. To describe them explicitly, in the notation of §4, we first define elements $x_{i,j}(m), x'_{i,j}(m) \in \mathrm{Dist}(G)$ for $m \geq 1$, $1 \leq i, j \leq n$, by setting $x_{i,j}(1) = e_{i,j}$, $x'_{i,j}(1) = e'_{i,j}$, and

$$x_{i,j}(m) = \sum_{s=1}^{n}(e_{i,s}x_{s,j}(m-1) + (-1)^{m-1}e'_{i,s}x'_{s,j}(m-1)), \tag{8.1}$$

$$x'_{i,j}(m) = \sum_{s=1}^{n}(e_{i,s}x'_{s,j}(m-1) + (-1)^{m-1}e'_{i,s}x_{s,j}(m-1)) \tag{8.2}$$

for $m > 1$. The following commutation relations are noted in [25] (they are easily verified using induction on m):

$$\begin{aligned}
[e_{i,j}, x_{s,t}(m)] &= \delta_{j,s}x_{i,t}(m) - \delta_{i,t}x_{s,j}(m),\\
[e'_{i,j}, x_{s,t}(m)] &= (-1)^{m-1}\delta_{j,s}x'_{i,t}(m) - \delta_{i,t}x'_{s,j}(m),\\
[e_{i,j}, x'_{s,t}(m)] &= \delta_{j,s}x'_{i,t}(m) - \delta_{i,t}x'_{s,j}(m),\\
[e'_{i,j}, x'_{s,t}(m)] &= (-1)^{m-1}\delta_{j,s}x_{i,t}(m) + \delta_{i,t}x_{s,j}(m).
\end{aligned}$$

Sergeev's central elements are the elements

$$z_r := \sum_{i=1}^{n} x_{i,i}(2r-1), \tag{8.3}$$

for $r \geq 1$. One checks directly using the above commutator relations that the z_r are indeed central. It is even proved in [25] that the elements $z_1, z_2, \dots$ *generate* the center of $\mathrm{Dist}(G)$ in case $k = \mathbb{C}$, but we will not need this fact. Instead, for $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i \in X(T)$, define the integer

$$z_r(\lambda) := \sum (-2)^{s-1}\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_s}(\lambda_{i_1}^2 - \lambda_{i_1})^{a_1}(\lambda_{i_2}^2 - \lambda_{i_2})^{a_2}\dots(\lambda_{i_s}^2 - \lambda_{i_s})^{a_s},$$

where the sum is over all $1 \leq s \leq r$, $1 \leq i_1 < i_2 < \dots < i_s \leq n$, and $a_1, a_2, \dots, a_s \geq 0$ with $a_1 + \dots + a_s = r - s$.

Lemma 8.4. *Let M be a $\mathrm{Dist}(G)$-supermodule and $v \in M_\lambda$ be a vector of weight λ annihilated by all $e_{i,j}, e'_{i,j}$ for all $1 \leq i < j \leq n$. Then, $z_r v = z_r(\lambda)v$.*

Proof. Let Ω be the left superideal of $\mathrm{Dist}(G)$ generated by all $e_{i,j}$ and $e'_{i,j}$ with $i < j$. We will write $\equiv$ for congruence modulo Ω throughout the proof. We first show that

$$x_{i,j}(m) \equiv x'_{i,j}(m) \equiv 0 \quad \text{for each } 1 \leq i < j \leq n \text{ and } m \geq 1. \tag{8.5}$$

This follows by induction on m, the induction base being clear. For $m > 1$, using the inductive assumption and the commutator relations, we get

$$\begin{aligned}
x_{i,j}(m) &= \sum_{s=1}^{n}(e_{i,s}x_{s,j}(m-1) + (-1)^{m-1}e'_{i,s}x'_{s,j}(m-1)) \\
&\equiv \sum_{s=j}^{n}(e_{i,s}x_{s,j}(m-1) + (-1)^{m-1}e'_{i,s}x'_{s,j}(m-1)) \\
&= \sum_{s=j}^{n}\big(x_{s,j}(m-1)e_{i,s} + x_{i,j}(m-1) \\
&\qquad -(-1)^{m-1}x'_{s,j}(m-1)e'_{i,s} - x_{i,j}(m-1)\big) \\
&\equiv 0.
\end{aligned}$$

The proof of (8.5) for $x'_{i,j}(m)$ is similar.

Now let $m \geq 3$ be an odd integer. We claim

$$x_{i,i}(m) \equiv (h_i^2 - h_i)x_{i,i}(m-2) - 2\sum_{s=i+1}^{n} h_i x_{s,s}(m-2). \tag{8.6}$$

Indeed, using (8.5) and the commutator relations, we see that

$$\begin{aligned}
x_{i,i}(m) &= \sum_{s=1}^{n}(e_{i,s}x_{s,i}(m-1) + e'_{i,s}x'_{s,i}(m-1)) \\
&\equiv h_i x_{i,i}(m-1) + h'_i x'_{i,i}(m-1) \\
&\quad + \sum_{s=i+1}^{n}(e_{i,s}x_{s,i}(m-1) + e'_{i,s}x'_{s,i}(m-1)) \\
&\equiv h_i x_{i,i}(m-1) + h'_i x'_{i,i}(m-1).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&x_{i,i}(m-1) \equiv h_i x_{i,i}(m-2) - h'_i x'_{i,i}(m-2) - 2\sum_{s=i+1}^{n} x_{s,s}(m-2), \\
&x'_{i,i}(m-1) \equiv h_i x'_{i,i}(m-2) - h'_i x_{i,i}(m-2).
\end{aligned}$$

Substituting these formulas into the above expression for $x_{i,i}(m)$ gives (8.6).

Now let $r \geq 1$ and $y_i := h_i^2 - h_i$. The theorem follows at once from the following formula:

$$x_{i,i}(2r-1) \equiv \sum (-2)^{s-1} h_{i_1}h_{i_2}\dots h_{i_s} y_{i_1}^{a_1} y_{i_2}^{a_2}\dots y_{i_s}^{a_s}, \tag{8.7}$$

where the sum is over all $1 \leq s \leq r,\ i = i_1 < i_2 < \dots < i_k \leq n$, and $a_1, a_2, \dots, a_s \geq 0$ with $a_1 + \dots + a_s = r - s$. To prove this, apply induction on r, the induction base being clear. For the induction step, let $r > 1$. Then by (8.6) and the inductive hypothesis, we have

$$\begin{aligned} x_{i,i}(2r-1) &\equiv (h_i^2 - h_i)x_{i,i}(2r-3) - 2\sum_{j=i+1}^{n} h_i x_{j,j}(2r-3) \\ &\equiv \sum_{s=1}^{r-1}(-2)^{s-1}\sum h_{i_1}h_{i_2}\dots h_{i_s} y_{i_1}^{a_1+1} y_{i_2}^{a_2}\dots y_{i_s}^{a_s} \\ &\quad + \sum_{j=i+1}^{n}\sum_{s=1}^{r-1}(-2)^s \sum h_i h_{j_1} h_{j_2}\dots h_{j_s} y_{j_1}^{b_1} y_{j_2}^{b_2}\dots y_{j_s}^{b_s}, \end{aligned}$$

where the first unmarked sum is over all $i = i_1 < i_2 < \cdots < i_s \le n$ and all $a_1, a_2, \dots, a_s \ge 0$ with $a_1 + \cdots + a_s = r - 1 - s$ and the second one is over all $j = j_1 < j_2 < \cdots < j_s \le n$ and all $b_1, b_2, \dots, b_s \ge 0$ with $b_1 + \cdots + b_s = r - 1 - s$. The formula (8.7) follows. $\square$

Let $\ell = (p-1)/2$, or $\ell = \infty$ in case $p = 0$. Given $j \in \mathbb{Z}$, define its *residue* $\mathrm{res}(j)$ to be the unique integer $r \in \{0, 1, \dots, \ell\}$ such that $j^2 - j \equiv r^2 + r \pmod p$. So:

$$\mathrm{res}(i) = \mathrm{res}(j) \text{ if and only if } (i^2 - i) \equiv (j^2 - j) \pmod p. \tag{8.8}$$

Now let $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X(T)$. For $r \in \{0, 1, \dots, \ell\}$, define

$$\begin{aligned} c_r \;:=\; & |\{(i,j) \mid 1 \le i \le n,\ 0 < j \le \lambda_i,\ \mathrm{res}(j) = r\}| \\ & - |\{(i,j) \mid 1 \le i \le n,\ \lambda_i < j \le 0,\ \mathrm{res}(j) = r\}|. \end{aligned}$$

Define the *content* $\mathrm{cont}(\lambda)$ of λ to be the tuple $(c_0, c_1, \dots, c_\ell)$. For example, let $n = 6$, $\lambda = (9, 7, 4, 0, -5, -8)$ and $p = 5$. In the following picture the nodes (i,j) are represented by boxes with the corresponding residues written in them:

$$\begin{array}{c|cccccccc|ccccccccc|c}
 & & & & & & & & & 0 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & \\
 & & & & & & & & & 0 & 1 & 2 & 1 & 0 & 0 & 1 & & & \\
- & & & & & & & & & 0 & 1 & 2 & 1 & & & & & & + \\
 & & & & & & & & & & & & & & & & & & \\
 & & & & & 0 & 1 & 2 & 1 & 0 & & & & & & & & & \\
 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & & & & & & & & & &
\end{array}$$

Hence, the content of λ is $(2, 4, 1)$. Also for $r, s \in \{0, 1, \dots, \ell\}$, we set

$$\begin{aligned} A_r(\lambda) &= |\{i \mid 1 \le i \le n,\ \mathrm{res}(\lambda_i + 1) = r\}|, \\ B_r(\lambda) &= |\{i \mid 1 \le i \le n,\ \mathrm{res}(\lambda_i) = r\}|, \\ d_{r,s}(\lambda) &= |\{i \mid 1 \le i \le n,\ \mathrm{res}(\lambda_i) = r,\ \mathrm{res}(\lambda_i + 1) = s\}|. \end{aligned}$$

Lemma 8.9. *Let $\lambda, \mu \in X(T)$ with $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i$. Then the following are equivalent:*

(i) $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$;
(ii) $d_{r,r+1}(\lambda) - d_{r+1,r}(\lambda) = d_{r,r+1}(\mu) - d_{r+1,r}(\mu)$ *for all* $r \in \{0, 1, \dots, \ell - 1\}$;
(iii) $A_r(\lambda) - B_r(\lambda) = A_r(\mu) - B_r(\mu)$ *for each* $r \in \{0, 1, \dots, \ell\}$;
(iv) $z_r(\lambda) \equiv z_r(\mu) \pmod p$ *for all* $r \ge 1$.

Proof. (i)$\Leftrightarrow$(ii). Let us write $\lambda\#\mu$ for the $2n$-tuple $(\lambda_1, \dots, \lambda_n, -\mu_1, \dots, -\mu_n)$. Observe that $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$ if and only if $\mathrm{cont}(\lambda\#\mu) = (0, 0, \dots, 0)$, and

$$d_{r,r+1}(\lambda) - d_{r+1,r}(\lambda) = d_{r,r+1}(\mu) - d_{r+1,r}(\mu)$$

if and only if

$$d_{r,r+1}(\lambda\#\mu) - d_{r+1,r}(\lambda\#\mu) = 0.$$

In other words, it is sufficient to prove the equivalence of (i) and (ii) in the special case $\mu = 0$.

So suppose $\mu = 0$ and let $\mathrm{cont}(\lambda) = (c_0, \dots, c_\ell)$. We need to show that $c_0 = \cdots = c_\ell = 0$ if and only if $d_{r,r+1}(\lambda) - d_{r+1,r}(\lambda) = 0$ for each $r \in \{0, 1, \dots, \ell - 1\}$, which follows easily from the equations:

$$\begin{aligned}
&c_0 + c_1 + \cdots + c_\ell = 0,\\
&c_r - c_{r+1} = d_{r,r+1}(\lambda) - d_{r+1,r}(\lambda) \quad \text{for } r \in \{0, 1, \dots, \ell - 2\},\\
&c_{\ell-1} - 2c_\ell = d_{\ell-1,\ell}(\lambda) - d_{\ell,\ell-1}(\lambda).
\end{aligned}$$

(ii)$\Leftrightarrow$(iii). This follows from the equations

$$\begin{aligned}
&A_0 + A_1 + \cdots + A_\ell = B_0 + B_1 + \cdots + B_\ell = n,\\
&A_0 = d_{0,0} + d_{1,0}, \quad B_0 = d_{0,0} + d_{0,1},\\
&A_r = d_{r-1,r} + d_{r+1,r}, \quad B_r = d_{r,r-1} + d_{r,r+1} \quad \text{for } r \in \{1, \dots, \ell - 1\}.
\end{aligned}$$

(iii)$\Leftrightarrow$(iv). Define

$$G_\lambda(t) := 1 - 2\sum_{r\geq 1} z_r(\lambda)t^r \in k[[t]].$$

We need to prove that given $\lambda, \mu \in X(T)$ with $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i$, we have that $G_\lambda(t) = G_\mu(t)$ if and only if $A_r(\lambda) - B_r(\lambda) = A_r(\mu) - B_r(\mu)$ for each $r = 0, 1, \dots, \ell$. We represent $G_\lambda(t)$ as a rational function:

$$\begin{aligned}
G_\lambda(t) &= \prod_{i=1}^n (1 - 2\sum_{r\geq 1} \lambda_i(\lambda_i^2 - \lambda_i)^{r-1}t^r)\\
&= \prod_{i=1}^n (1 - 2\lambda_i t(1 - (\lambda_i^2 - \lambda_i)t)^{-1})\\
&= \prod_{i=1}^n \frac{1 - (\lambda_i^2 - \lambda_i)t - 2\lambda_i t}{1 - (\lambda_i^2 - \lambda_i)t}\\
&= \prod_{i=1}^n \frac{1 - (\lambda_i^2 + \lambda_i)t}{1 - (\lambda_i^2 - \lambda_i)t}.
\end{aligned}$$

Counting multiplicities of zeros and poles, we see that $G_\lambda(t) = G_\mu(t)$ if and only if $a_r(\lambda) - b_r(\lambda) = a_r(\mu) - b_r(\mu)$ for each $r = 0, 1, \dots, p - 1$ where

$$\begin{aligned}
a_r(\lambda) &= |\{i = 1, \dots, n \mid (\lambda_i + 1)^2 - (\lambda_i + 1) \equiv r \pmod p\}|,\\
b_r(\lambda) &= |\{i = 1, \dots, n \mid \lambda_i^2 - \lambda_i \equiv r \pmod p\}|.
\end{aligned}$$

Finally using (8.8), we have that $a_r(\lambda) - b_r(\lambda) = a_r(\mu) - b_r(\mu)$ for each $r = 0, \dots, p-1$ if and only if $A_r(\lambda) - B_r(\lambda) = A_r(\mu) - B_r(\mu)$ for each $r = 0, 1, \dots, \ell$. $\square$

Now we obtain the main result of the section, being an analogue of the "linkage principle". The linkage principle in case $p = 0$ was proved originally by Penkov; see [18] for the shorter statement and proof in that case.

Theorem 8.10. *Let $\lambda \in X_p^+(T)$. All composition factors of $H^0(\lambda)$ are of the form $L(\mu)$ for $\mu \leq \lambda$ with $\mathrm{cont}(\mu) = \mathrm{cont}(\lambda)$.*

Proof. It suffices to show that $(z_r - z_r(\lambda))$ annihilates $H^0(\lambda)$. By Lemma 6.10(ii), the B^+-socle of $H^0(\lambda)$ equals $H^0(\lambda)_\lambda$. So by Lemma 8.4, $(z_r - z_r(\lambda))$ annihilates $H^0(\lambda)_\lambda$. Therefore $(z_r - z_r(\lambda))H^0(\lambda)$ intersects the B^+-socle $H^0(\lambda)_\lambda$ trivially, hence is zero. □

Applying Theorem 7.5 (or an obvious direct argument) gives:

Corollary 8.11. *For $\lambda, \mu \in X_p^+(T)$ with $\mathrm{cont}(\lambda) \neq \mathrm{cont}(\mu)$, we have*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0.$$

9. Steinberg's tensor product theorem

Suppose throughout the section that $p \neq 0$. We wish next to prove the analogue for $G = Q(n)$ of the Steinberg tensor product theorem [27], following the approach of [4]. We will often now appeal to well-known results about $G_{\mathrm{ev}} = GL(n)$. For instance, the coordinate ring $k[G_{\mathrm{ev}}]$ is the localization of the free polynomial algebra $k[c_{i,j}\,|1 \leq i, j \leq n]$ in n^2 even indeterminates at determinant. We have the subgroups $T = H_{\mathrm{ev}}, B_{\mathrm{ev}}, B_{\mathrm{ev}}^+, U_{\mathrm{ev}}, U_{\mathrm{ev}}^+$ of G_{ev}, and the algebra of distributions $\mathrm{Dist}(G_{\mathrm{ev}})$ was described as a subalgebra of $\mathrm{Dist}(G)$ in (5.3). For every $\lambda \in X^+(T)$, we have an irreducible G_{ev}-module of highest weight λ which we denote by $L_{\mathrm{ev}}(\lambda)$; it can be defined as the simple socle of the induced module $H^0_{\mathrm{ev}}(\lambda) = \mathrm{ind}_{B_{\mathrm{ev}}}^{G_{\mathrm{ev}}} k_\lambda$, see [11].

For $r \geq 1$, we have the *Frobenius morphism* $F^r : G \to G_{\mathrm{ev}}$ defined for a commutative superalgebra A by letting $F^r : G(A) \to G_{\mathrm{ev}}(A)$ be the group homomorphism obtained by raising matrix entries to the p^r-th power (note $a^{p^r} = 0$ for all $a \in A_{\bar 1}$ so this makes sense). Clearly F^r stabilizes the various subgroups H, B, B^+, U, U^+ of G, giving us morphisms also all denoted by F^r from each of these supergroups to their even part. We denote the kernel of F^r by G_r (resp. $H_r, B_r, B_r^+, U_r, U_r^+$). Then, G_r is a normal subgroup of G, called the *rth Frobenius kernel.*

Lemma 9.1. *$F^r : G \to G_{\mathrm{ev}}$ is a quotient of G by G_r in the category of superschemes, that is, for any morphism of superschemes $f : G \to X$ that is constant on $G_r(A)$-cosets in $G(A)$ (for each superalgebra A) there is a unique $\tilde f : G_{\mathrm{ev}} \to X$ such that $f = \tilde f \circ F^r$.*

Proof. Let $\sigma : G \to G_{\mathrm{ev}}$ be the morphism defined for each A as the projection $G(A) \to G_{\mathrm{ev}}(A)$ onto the diagonal, see (3.1). Let $f : G \to X$ be a morphism of superschemes constant on G_r-cosets. For any superalgebra A and any element $g \in G(A)$ of the form (3.1), we have that

$$\begin{pmatrix} I & S'S^{-1} \\ -S'S^{-1} & I \end{pmatrix}^{-1} \begin{pmatrix} S & S' \\ -S' & S \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix},$$

i.e. $\sigma(g) = hg$ for some $h \in G_r(A)$. Hence, $f = (f|_{G_{\mathrm{ev}}}) \circ \sigma$. But $F^r|_{G_{\mathrm{ev}}} : G_{\mathrm{ev}} \to G_{\mathrm{ev}}$ is the quotient of G_{ev} by $G_{r,\mathrm{ev}}$ by the purely even theory [11, I.9.5]. Since $f|_{G_{\mathrm{ev}}}$ is constant on $G_{r,\mathrm{ev}}$-cosets in G_{ev}, we get a unique morphism $\tilde{f} : G_{\mathrm{ev}} \to X$ such that $f|_{G_{\mathrm{ev}}} = \tilde{f} \circ (F^r|_{G_{\mathrm{ev}}})$. Hence $f = \tilde{f} \circ (F^r|_{G_{\mathrm{ev}}}) \circ \sigma = \tilde{f} \circ F^r$. $\square$

Note that $k[G_r] \cong k[G]/I_{p^r}$ where

$$I_{p^r} = \langle s_{i,j}^{p^r},\ s_{l,l}^{p^r} - 1 \mid 1 \leq i \neq j \leq n,\ 1 \leq l \leq n\rangle.$$

So a basis of $k[G_r]$ is given by images of the monomials

$$\prod_{1\leq i<j\leq n} s_{j,i}^{a_{j,i}} \left(s'_{j,i}\right)^{d_{j,i}} \cdot \prod_{1\leq l\leq n} s_{l,l}^{a_{l,l}} \left(s'_{l,l}\right)^{d_{l,l}} \cdot \prod_{1\leq i<j\leq n} s_{i,j}^{a_{i,j}} \left(s'_{i,j}\right)^{d_{i,j}}$$

for integers $a_{i,j} \in \{0, 1, \ldots, p^r - 1\}, d_{i,j} \in \{0, 1\}$, where the products are taken in any fixed order. In particular, $\dim k[G_r] = (2p^r)^{n^2}$, and G_r is a *finite* algebraic supergroup (cf. [11, I.8.1]). Moreover, the p^r-th power of each generator of the augmentation ideal $I_e = \ker E$ of $k[G]$ (i.e. the elements $s_{i,j}$, $s'_{i,j}$, $s_{r,r} - 1$, $s'_{r,r}$ for $1 \leq i \neq j \leq n$, $1 \leq r \leq n$) is an element of I_{p^r}, hence the image of I_e in $k[G_r]$ is a nilpotent ideal. So $\mathrm{Dist}(G_r)$ can actually be identified with the dual superalgebra $k[G_r]^*$. It follows easily (see [11, I.8.3] and the discussion at the beginning of [2, §5]) that the category of G_r-supermodules is isomorphic to the category of $\mathrm{Dist}(G_r)$-supermodules. As a basis for $\mathrm{Dist}(G_r) \subset \mathrm{Dist}(G)$, we can take the ordered PBW monomials

$$\prod_{1\leq i<j\leq n} e_{j,i}^{(a_{j,i})} \left(e'_{j,i}\right)^{d_{j,i}} \cdot \prod_{1\leq l\leq n} \binom{h_l}{a_{l,l}} \left(h'_l\right)^{d_{l,l}} \cdot \prod_{1\leq i<j\leq n} e_{i,j}^{(a_{i,j})} \left(e'_{i,j}\right)^{d_{i,j}}$$

for integers $a_{i,j} \in \{0, 1, \ldots, p^r - 1\}$, $d_{i,j} \in \{0, 1\}$. Similarly, one can describe explicit bases for $k[B_r]$, $k[U_r]$, $\mathrm{Dist}(B_r)$, $\mathrm{Dist}(U_r)$ etc. We obviously have:

Lemma 9.2. $\mathrm{Dist}(G_r)$ *is a free right* $\mathrm{Dist}(B_r^+)$*-supermodule with basis given by the ordered monomials*

$$\Big\{ \prod_{1\leq i<j\leq n} e_{j,i}^{(a_{j,i})}(e'_{j,i})^{d_{j,i}} \mid a_{j,i} \in \{0, 1, \ldots, p^r - 1\},\ d_{j,i} \in \{0, 1\}\Big\}.$$

Identifying G_r- (resp. B_r^+-) supermodules with $\mathrm{Dist}(G_r)$- (resp. $\mathrm{Dist}(B_r^+)$-) supermodules, we have the *coinduction functor*

$$\mathrm{coind}_{B_r^+}^{G_r} : \mathfrak{mod}_{B_r^+} \to \mathfrak{mod}_{G_r}, \quad \mathrm{coind}_{B_r^+}^{G_r} ? = \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r^+)} ?.$$

Lemma 9.2 gives that this is an exact functor that is left adjoint to $\mathrm{res}_{B_r^+}^{G_r}$. Indeed, given any B_r^+-supermodule M, we have that

$$\mathrm{coind}_{B_r^+}^{G_r} M \simeq \mathrm{Dist}(U_r) \otimes M \tag{9.3}$$

as a vector superspace, so in particular $\mathrm{coind}_{B_r^+}^{G_r} M$ has dimension $(2p^r)^N \dim M$ for finite dimensional M, where $N = n(n-1)/2$.

Now consider the representation theory of H_r. For $\lambda \in X(T)$, let $\mathfrak{u}_r(\lambda)$ denote the restriction of the H-supermodule $\mathfrak{u}(\lambda)$ to H_r. Arguing in the same way as for Lemma 6.4, one shows:

Lemma 9.4. *The modules $\{\mathfrak{u}_r(\lambda) \mid \lambda \in X(T)\}$ give a complete set of irreducible H_r-supermodules. Moreover, for $\lambda, \mu \in X(T)$, we have $\mathfrak{u}_r(\lambda) \cong \mathfrak{u}_r(\mu)$ if and only if $\lambda - \mu \in p^r X(T)$.*

We also write $\mathfrak{u}_r(\lambda)$ for its inflation to B_r (resp. B_r^+). By the lemma, the modules $\{\mathfrak{u}_r(\lambda) \mid \lambda \in X(T)\}$ also give a complete set of irreducible B_r- (resp. B_r^+-) supermodules. Given $\lambda \in X(T)$, let

$$Z_r(\lambda) := \operatorname{coind}_{B_r^+}^{G_r} \mathfrak{u}_r(\lambda).$$

Note $\dim Z_r(\lambda) = 2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor}(2p^r)^N$. Let $L_r(\lambda)$ be the G_r-head of $Z_r(\lambda)$.

Theorem 9.5. *The modules $\{L_r(\lambda) \mid \lambda \in X(T)\}$ form a complete set of irreducible G_r-supermodules, with $L_r(\lambda) \cong L_r(\mu)$ if and only if $\lambda - \mu \in p^r X(T)$.*

Proof. One can easily see that the B_r-module $\mathfrak{u}_r(\lambda)$ is a quotients of $Z_r(\lambda)$. Moreover, by (9.3), we have $\dim \operatorname{Hom}_{U_r}(Z_r(\lambda), k) = \dim \mathfrak{u}_r(\lambda)$. Hence, $Z_r(\lambda)$ has irreducible B_r-head $\simeq \mathfrak{u}_r(\lambda)$. It follows at once that $Z_r(\lambda)$ has irreducible G_r-head, so that $L_r(\lambda)$ is an irreducible G_r-supermodule. Now, if L is any irreducible G_r-supermodule, choose $\lambda \in X(T)$ so that $\operatorname{Hom}_{B_r^+}(\mathfrak{u}_r(\lambda), L) \neq 0$. Then Frobenius reciprocity gives that L is isomorphic to a quotient of $Z_r(\lambda)$, hence $L \cong L_r(\lambda)$. Finally, since the B_r-head of $L_r(\lambda)$ is $\mathfrak{u}_r(\lambda)$, we get from Lemma 9.4 that $L_r(\lambda) \cong L_r(\mu)$ if and only if $\lambda - \mu \in p^r X(T)$. $\square$

Now we begin the proof of the Steinberg tensor product theorem. It suffices from now on to consider the special case $r = 1$, since the tensor product theorem is already proved for G_{ev}.

Lemma 9.6. *Let L be an irreducible G-supermodule. Then, L is completely reducible as a G_1-supermodule.*

Proof. Pick L_1 in the G_1-socle of L. Since G_1 is a normal subgroup of G, we have for each $g \in G(k)$ that the translate gL_1 is an irreducible G_1-submodule of L. Hence $M := \sum_{g \in G(k)} gL_1$ is a completely reducible G_1-submodule of L. To complete the proof, we just need to show that $M = L$. This will follow from irreducibility of L if we check that M is invariant under the action of $\mathrm{Dist}(G)$. By definition, M is invariant under $G(k)$. So M is a G_{ev}-submodule of L, for example by [11, I.2.12(5)] since $G(k) = G_{\mathrm{ev}}(k)$ is dense in G_{ev} (cf [11, I.6.16]). So M is invariant under both $\mathrm{Dist}(G_{\mathrm{ev}})$ and $\mathrm{Dist}(G_1)$. But $\mathrm{Dist}(G)$ is generated by $\mathrm{Dist}(G_{\mathrm{ev}})$ and $\mathrm{Dist}(G_1)$, so M is invariant under $\mathrm{Dist}(G)$. $\square$

Lemma 9.7. *Let $\lambda \in X_p^+(T)$. Then,* $\mathrm{Dist}(G_1)L(\lambda)_\lambda$ *is a G_1-submodule of $L(\lambda)$ isomorphic to $L_1(\lambda)$.*

Proof. As a B^+-supermodule, $L(\lambda)_\lambda \simeq \mathfrak{u}(\lambda)$, so as a B_1^+-supermodule it is $\simeq \mathfrak{u}_1(\lambda)$. Hence there is an even B_1^+-supermodule homomorphism $\mathfrak{u}_1(\lambda) \to L(\lambda)$ with image $L(\lambda)_\lambda$. Applying Frobenius reciprocity, we obtain a G_1-supermodule homomorphism $Z_1(\lambda) \to L(\lambda)$ with image $\mathrm{Dist}(G_1)L(\lambda)_\lambda$. By Lemma 9.6, $\mathrm{Dist}(G_1)L(\lambda)_\lambda$ is completely reducible, while $Z_1(\lambda)$ has irreducible G_1-head. So in fact $\mathrm{Dist}(G_1)L(\lambda)_\lambda$ is an irreducible G_1-supermodule isomorphic to the head $L_1(\lambda)$ of $Z_1(\lambda)$. $\square$

Now we introduce the *restricted* weights (cf. [2, §9]): call $\lambda = \sum_{i=1}^n \lambda_i\varepsilon_i \in X_p^+(T)$ *restricted* if $\lambda_i - \lambda_{i+1} \leq p$ when $p \nmid \lambda_i$, and $\lambda_i - \lambda_{i+1} < p$ when $p \mid \lambda_i$, for each $i = 1, \dots, n-1$. Let $X_p^+(T)_{\mathrm{res}}$ denote the set of all restricted $\lambda \in X_p^+(T)$. The proof of the next lemma is based on the argument in [1, 6.4].

Lemma 9.8. *For $\lambda \in X_p^+(T)_{\mathrm{res}}$, the irreducible G-supermodule $L(\lambda)$ is also irreducible as a G_1-supermodule, and $\mathrm{res}^G_{G_1} L(\lambda) \simeq L_1(\lambda)$.*

Proof. Let $M = \mathrm{Dist}(G_1)L(\lambda)_\lambda$. By Lemma 9.7, $M \simeq L_1(\lambda)$. So we just need to show that $M = L(\lambda)$, which will follow if we can show that M is invariant under the action of $\mathrm{Dist}(G)$. But $\mathrm{Dist}(G)$ is generated by $\mathrm{Dist}(G_{\mathrm{ev}})$ and $\mathrm{Dist}(G_1)$, so we just need to check that M is invariant under the action of $\mathrm{Dist}(G_{\mathrm{ev}})$. Since B_{ev}^+ normalizes G_1 and $L(\lambda)_\lambda$ is a B_{ev}^+-submodule of $L(\lambda)$, M is invariant under the action of $\mathrm{Dist}(B_{\mathrm{ev}}^+)$; in particular, M is the sum of its weight spaces. Therefore we just need to prove that:

(∗) $e^{(r)}_{i+1,i}v \in M$ for all $v \in M_\mu$, $i = 1, \dots, n-1$, $r \geq 1$ and all $\mu \in X(T)$.

We prove (∗) by downward induction on the weight μ.

To start the induction, we need to consider the case $\mu = \lambda$. Here we need to show that $e^{(r)}_{i+1,i}v \in M$ for all $r \geq 1$ and $v \in L(\lambda)_\lambda$. But $e^{(r)}_{i+1,i}v = 0$ for $r > \lambda_i - \lambda_{i+1}$ by $SL(2)$ theory, while clearly $e^{(r)}_{i+1,i}v \in M$ for $r < p$. This just leaves the case when $\lambda_i - \lambda_{i+1} = p$, when $p \nmid \lambda_i$ by assumption; we need to check that $e^{(p)}_{i+1,i}v \in M$ for all $v \in L(\lambda)_\lambda$. In fact we will show more, namely, that

$$e^{(p)}_{i+1,i}v = e^{(p-1)}_{i+1,i}e'_{i+1,i}hv, \quad \text{where} \quad h = \frac{h'_i - h'_{i+1}}{\lambda_i + \lambda_{i+1}}.$$

Well, consider $m = e^{(p)}_{i+1,i}v - e^{(p-1)}_{i+1,i}e'_{i+1,i}hv$. If $m \neq 0$, then (since $L(\lambda)^{U^+} = L(\lambda)_\lambda$) we can find a PBW monomial x in $\mathrm{Dist}(U^+)$ such that xm is a non-zero vector in $L(\lambda)_\lambda$. But by weights, the only possibilities for the monomial x are $e^{(p)}_{i,i+1}$ and $e'_{i,i+1}e^{(p-1)}_{i,i+1}$. We note that $(h'_i - h'_{i+1})hv = v$. So using the commutator relations (cf. [26, Lemma 5]), we get that:

$$\begin{aligned}
e^{(p)}_{i,i+1}m &= \binom{h_i - h_{i+1}}{p}v - \binom{h_i - h_{i+1} - 1}{p-1}e_{i,i+1}e'_{i+1,i}hv \\
&= v - (h'_i - h'_{i+1})hv = 0, \\
e'_{i,i+1}e^{(p-1)}_{i,i+1}m &= e'_{i,i+1}e_{i+1,i}\binom{h_i - h_{i+1} - 1}{p-1}v \\
&\quad - e'_{i,i+1}\binom{h_i - h_{i+1}}{p-1}e'_{i+1,i}hv \\
&\quad - e'_{i,i+1}e_{i+1,i}\binom{h_i - h_{i+1} - 2}{p-2}e_{i,i+1}e'_{i+1,i}hv \\
&= (h'_i - h'_{i+1})v - 0 - (h'_i - h'_{i+1})^2hv = 0.
\end{aligned}$$

This shows that $m = 0$, as required.

Now take $\mu < \lambda$ and assume $(*)$ has been proved for all greater μ. Any $v \in M_\mu$ can be written as $e_{s,s+1}w$ or $e'_{s,s+1}w$ for some s and $w \in M_{\mu+\varepsilon_s-\varepsilon_{s+1}}$. Then,

$$e^{(r)}_{i,i+1}v = e^{(r)}_{i,i+1}e_{s,s+1}w \quad \text{or} \quad e^{(r)}_{i,i+1}e'_{s,s+1}w.$$

Now one uses the commutator relations once more together with the inductive hypothesis to see that in either case the expression on the right hand side lies in M. $\square$

Given any G_{ev}-module M (viewed as a supermodule concentrated in degree $\bar{0}$) we can inflate through $F = F^1 : G \to G_{\text{ev}}$ to obtain a G-supermodule F^*M, the *Frobenius twist* of M. Thus, we have a functor

$$F^* : \mathfrak{mod}_{G_{\text{ev}}} \to \mathfrak{mod}_G.$$

Conversely, given a G-supermodule N, there is an induced G_{ev}-module structure on the fixed point space N^{G_1}: the representation $G \to GL(N^{G_1})$ is constant on G_1-cosets so factors to $G_{\text{ev}} \to GL(N^{G_1})$ by Lemma 9.1. (In case N is infinite dimensional, equal to the direct limit of its finite dimensional submodules, one needs here to use the fact that taking fixed points commutes with direct limits.) Thus we have a functor

$$?^{G_1} : \mathfrak{mod}_G \to \mathfrak{mod}_{G_{\text{ev}}}$$

which is right adjoint to F^*. Now we can prove the main result of the section:

Theorem 9.9. *For $\lambda \in X^+_p(T)_{\text{res}}$, $\mu \in X^+(T)$,*

$$L(\lambda + p\mu) \cong L(\lambda) \otimes F^*L_{\text{ev}}(\mu)$$

as a G-supermodule.

Proof. For $\lambda \in X^+_p(T)_{\text{res}}$, $L(\lambda)$ is irreducible as a G_1-supermodule by Lemma 9.8. By Lemma 9.7,

$$H := \operatorname{Hom}_{G_1}(L(\lambda), L(\lambda + p\mu))_{\bar{0}}$$

is non-zero (replacing $L(\lambda + p\mu)$ by $\Pi L(\lambda + p\mu)$ if necessary). In what follows, we view H as a G-supermodule by conjugation, so the action of $u \in \operatorname{Dist}(G)$ is given by $(uf)(l) = \sum_i u_i f(\sigma(u'_i)l)$ for $f \in H, l \in L(\lambda)$ if $\delta(u) = \sum_i u_i \otimes u'_i$ (where σ is the antipode on $\operatorname{Dist}(G)$). One checks directly from this that the map

$$\theta : H \otimes L(\lambda) \to L(\lambda + p\mu), \quad f \otimes l \mapsto f(l)$$

is an even G-supermodule homomorphism. Since it is non-zero and $L(\lambda + p\mu)$ is irreducible, θ is surjective. On the other hand, by Schur's lemma,

$$\begin{aligned} & \dim \operatorname{Hom}_{G_1}(L(\lambda), L(\lambda + p\mu))_{\bar{0}} \otimes L(\lambda) \\ \leq \; & (\dim L(\lambda + p\mu)/\dim L(\lambda)) \dim L(\lambda) \\ = \; & \dim L(\lambda + p\mu), \end{aligned}$$

hence θ is in fact an isomorphism. Finally, since the action of G_1 on H is trivial, we must have that $H \simeq F^*M$ for some G_{ev}-module M. Moreover, M must be irreducible since $L(\lambda + p\mu)$ is irreducible. But the highest weight of H is $p\mu$, hence $M \cong L_{\text{ev}}(\mu)$. $\square$

10. Polynomial representations

Recall that $k[Mat] = k[s_{i,j}, s'_{i,j} \mid 1 \leq i,j \leq n]$, and $k[G]$ is the localization of $k[Mat]$ at det. We call a representation M of $G = Q(n)$ *polynomial* if the image of the structure map $\eta : M \to M \otimes k[G]$ lies in $M \otimes k[Mat]$.

More generally, given any standard Levi subgroup G_γ of G, we can talk about polynomial representations of G_γ, defined in an entirely analogous way. In particular, we can talk about polynomial representations of H: an H-supermodule M with structure map $\eta : M \to M \otimes k[H]$ is polynomial if $\eta(M) \subseteq M \otimes k[x_i, x'_i \mid 1 \leq i \leq n]$ where x_i, x'_i are the restrictions of the coordinate functions $s_{i,i}, s'_{i,i}$ to H respectively.

The natural G-supermodule V is a polynomial representation, as is any subquotient of $V^{\otimes d}$ for any $d \geq 0$, but the dual G-supermodule V^* is not polynomial. Thus, the natural duality $*$ on finite dimensional G-supermodules does not respect polynomial representations. However, there is another duality denoted τ which does take polynomial representations to polynomial representations. To define this, let $\tau : k[G] \to k[G]$ be the unique linear map which maps $1 \mapsto 1, s_{i,j} \mapsto s_{j,i}, s'_{i,j} \mapsto s'_{j,i}$ and satisfies $\tau(fg) = \tau(g)\tau(f)$ for all $f, g \in k[G]$. By checking on the generators $s_{i,j}, s'_{i,j}, \det^{-1}$, one verifies:

Lemma 10.1. $\Delta \circ \tau = T \circ (\tau \otimes \tau) \circ \Delta$ *where* $T : k[G] \otimes k[G] \to k[G] \otimes k[G]$ *is the unsigned twist* $f_1 \otimes f_2 \mapsto f_2 \otimes f_1$.

Now let M be a finite dimensional G-supermodule with structure map $\eta : M \to M \otimes k[G]$. Pick a basis $m_1, \dots, m_r$ for M and write $\eta(m_j) = \sum_{i=1}^r m_i \otimes c_{i,j}$ for $c_{i,j} \in k[G]$. Let $f_1, \dots, f_r$ be the basis for M^* dual to $m_1, \dots, m_r$. Define M^τ to be the G-supermodule equal to M^* as a vector superspace with structure map

$$\eta^\tau : M^\tau \to M^\tau \otimes k[G], \qquad f_j \mapsto \sum_{i=1}^r f_i \otimes \tau(c_{j,i}).$$

Note the definition of η^τ is independent of the choice of basis, and it is easily checked to be a comodule structure map using Lemma 10.1. Obviously, $(M^\tau)^\tau \simeq M$ and, since τ leaves $k[Mat]$ invariant, M^τ is polynomial if and only if M is polynomial. Moreover, M and M^τ have the same character since τ fixes each $s_{i,i}$.

Let

$$\Lambda(T) = \{\lambda = \textstyle\sum_{i=1}^n \lambda_i \varepsilon_i \in X(T) \mid \lambda_i \geq 0 \text{ for each } i = 1, \dots, n\}, \quad (10.2)$$
$$\Lambda_p^+(T) = \Lambda(T) \cap X_p^+(T). \quad (10.3)$$

If M is a polynomial representation of G, it is polynomial over H so in particular all its weights lie in $\Lambda(T)$. Hence all its composition factors are of the form $L(\lambda)$ for $\lambda \in \Lambda_p^+(T)$. We can state [2, Theorem 10.1] (see also [24] over $\mathbb{C}$) as follows:

Theorem 10.4. *The modules* $\{L(\lambda) \mid \lambda \in \Lambda_p^+(T)\}$ *give a complete set of pairwise non-isomorphic irreducible polynomial representations of* G.

Note however that there exist non-polynomial extensions of polynomial $L(\lambda)$'s, unlike the situation for $GL(n)$. For example, the G-supermodule $\widetilde{k}$ from (7.6) is an extension of two copies of the trivial G-supermodule that is not even polynomial over H. The goal in the remainder of the section is to prove that a G-supermodule is polynomial if and only if it is polynomial on restriction to H. There are several proofs of the analogous statement for $GL(n)$ in the literature, see [7], [8], [10, Prop.3.4] and [12, Theorem 5.3]. Of these, Jantzen's argument from [12] adapts easily to our situation. First we record the following lemma which will easily settle the case $p = 0$:

Lemma 10.5. *If $p = 0$, every representation of G that is polynomial on restriction to H is completely reducible.*

Proof. This is trivial to check directly if $G = Q(1)$. Hence, since H is a direct product of copies of $Q(1)$, every polynomial representation of H is completely reducible.

As observed above, if M is a polynomial representation of G that is polynomial over H then all its composition factors are of the form $L(\lambda)$ for $\lambda \in \Lambda_p^+(T)$. So take $\lambda, \mu \in \Lambda_p^+(T)$ with $\mu \not> \lambda$ and suppose we have an extension

$$0 \longrightarrow L(\lambda) \longrightarrow M \longrightarrow L(\mu) \longrightarrow 0$$

where M is a G-supermodule that is polynomial over H. To prove the lemma, it suffices to show that the extension splits. By the linkage principle in characteristic 0, $H^0(\lambda) = L(\lambda)$ for $\lambda \in \Lambda_p^+(T)$. Thus, $L(\lambda)$ is the induced module $\mathrm{ind}_B^G \mathfrak{u}(\lambda)$. Using this, one constructs a splitting $M \to L(\lambda)$ of the above short exact sequence in exactly the same way as in the proof that $\mathrm{Ext}^1_G(L(\mu), \widetilde{H}^0(\lambda)) = 0$ in Theorem 7.5, replacing the injectivity of $\widetilde{\mathfrak{u}}(\lambda)$ there with the complete reducibility of M as an H-supermodule. □

Now suppose for the next lemma that $p > 0$. Then we can consider the "thickened" Frobenius kernel G_rT defined to be the closed subgroup $(F^r)^{-1}T$ of G, so $(G_rT)(A) = G_r(A)T(A)$ for each superalgebra A. The coordinate ring $k[G_rT]$ is the quotient of $k[G]$ by the ideal generated by $\{s_{i,j}^{p^r} \mid i \neq j\}$. Note also that H is a subgroup of G_rT. On G_rT, we have that $\det^{p^r} = (s_{1,1} \dots s_{n,n})^{p^r}$, hence $\det^{-1} = \det^{p^r-1}(s_{1,1} \dots s_{n,n})^{-p^r}$. So

$$k[G_rT] = k[Mat][s_{1,1}^{-1}, \dots, s_{n,n}^{-1}]/(s_{i,j}^{p^r} \mid i \neq j). \tag{10.6}$$

We also define the following subsets of $k[G_rT]$:

$$R := k[Mat]/(s_{i,j}^{p^r} \mid i \neq j), \tag{10.7}$$

$$R_l := k[Mat][s_{1,1}^{-1}, \dots, s_{l-1,l-1}^{-1}, s_{l+1,l+1}^{-1}, \dots, s_{n,n}^{-1}]/(s_{i,j}^{p^r} \mid i \neq j), \tag{10.8}$$

for each $l = 1, \dots, n$. We call a G_rT-supermodule M with structure map $\eta : M \to M \otimes k[G_rT]$ *polynomial* if $\eta(M) \subseteq M \otimes R$.

Lemma 10.9. *If $p > 0$, every G_rT-supermodule that is polynomial on restriction to H is polynomial over G_rT.*

Proof. Let $u_{i,j}, u'_{i,j}$ $(i > j)$, $v_{i,j}, v'_{i,j}$ $(i < j)$ and x_i, x'_i $(1 \leq i \leq n)$ be the standard coordinate functions (restrictions of various $s_{i,j}, s'_{i,j}$) on U_r, U_r^+ and H respectively. Consider the morphism

$$\mu : U_r \times H \times U_r^+ \to G_rT$$

induced by multiplication. We claim that μ is an isomorphism of superschemes, and moreover the functions $u_{i,j}, u'_{i,j}, v_{i,j}, v'_{i,j}$ and x_i, x'_i belong to R_n when viewed as elements of $k[G_rT]$ via the isomorphism μ. This is proved by induction on n in a similar way to [12, Lemma 5.1]. For example, for $n = 2$, the matrix

$$g = \left[\begin{array}{cc|cc} a & b & a' & b' \\ c & d & c' & d' \\ \hline -a' & -b' & a & b \\ -c' & -d' & c & d \end{array}\right] \in (G_rT)(A)$$

can be factorized as

$$\begin{aligned} g \quad = \quad & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \frac{ca+c'a'}{a^2} & 1 & \frac{c'a-ca'}{a^2} & 0 \\ \hline 0 & 0 & 1 & 0 \\ \frac{ca'-c'a}{a^2} & 0 & \frac{ca+c'a'}{a^2} & 1 \end{array}\right] \times \left[\begin{array}{cc|cc} a & 0 & a' & 0 \\ 0 & q & 0 & q' \\ \hline -a' & 0 & a & 0 \\ 0 & -q' & 0 & q \end{array}\right] \\ & \times \left[\begin{array}{cc|cc} 1 & \frac{ab+a'b'}{a^2} & 0 & \frac{ab'-a'b}{a^2} \\ 0 & 1 & 0 & 0 \\ \hline 0 & \frac{a'b-ab'}{a^2} & 1 & \frac{ab+a'b'}{a^2} \\ 0 & 0 & 0 & 1 \end{array}\right], \end{aligned}$$

where

$$\begin{aligned} q &= d - \frac{abc + ab'c' + a'b'c - a'bc'}{a^2}, \\ q' &= d' - \frac{ab'c + a'b'c' + abc' - a'bc}{a^2}. \end{aligned}$$

The point is that d^{-1} does not appear in these expressions.

Now let M be a G_rT-supermodule that is polynomial over H, with structure map $\eta : M \to M \otimes k[G_rT]$. Let m_i $(i \in I)$ be a homogeneous basis for M and write

$$\eta(m_j) = \sum_{i \in I} m_i \otimes c_{i,j} \tag{10.10}$$

for some $c_{i,j} \in k[G_rT]$. Given $g \in (G_rT)(A)$ for some superalgebra A, write $g = uhv$ for $u \in U_r, h \in H, v \in U_r^+$. Then,

$$c_{i,j}(g) = \sum_{l,m} c_{i,l}(u) c_{l,m}(h) c_{m,j}(v).$$

Now, $c_{i,l}(u)$ and $c_{m,j}(v)$ are polynomial functions in the coordinates $u_{i,j}, u'_{i,j}$ on U_r and $v_{i,j}, v'_{i,j}$ of U_r^+ respectively. Moreover, by the assumption that M is polynomial over H, $c_{l,m}(h)$ is a polynomial in the coordinates x_i, x'_i on H. This shows each $c_{i,j} \in R_n$ thanks to the previous paragraph.

The symmetric group W acts on G_rT by conjugation, hence for each $w \in W$ we obtain a new G_rT-supermodule ${}^wM$ by twisting the action by w. The matrix coefficients of ${}^wM$ are the functions $w \cdot c_{i,j}$ where the $c_{i,j}$'s are as defined in (10.10) and $(w \cdot c_{i,j})(g) = c_{i,j}(w^{-1}gw)$. Clearly each ${}^wM$ is polynomial over H too, so applying the previous paragraph to ${}^wM$ instead shows that $w \cdot c_{i,j} \in R_n$, hence $c_{i,j} \in R_{w^{-1}n}$, for all $w \in W$. But $R = \bigcap_{l=1}^n R_l$. Thus we have shown that each $c_{i,j} \in R$, hence M is polynomial over G_rT. $\square$

Theorem 10.11. *A G-supermodule M is polynomial if and only if it is polynomial on restriction to H.*

Proof. Obviously, any polynomial representation of G is polynomial over H. So suppose instead that M is a G-supermodule that is polynomial over H. In case $p = 0$, Lemma 10.5 shows that M is completely reducible. Hence M is a direct sum of $L(\lambda)$'s for $\lambda \in \Lambda_p^+(T)$. But all such $L(\lambda)$ are polynomial by Theorem 10.4. Now assume $p > 0$. By Lemma 10.9, M is polynomial over G_rT for all $r \geq 1$. This implies that M is polynomial over G by the argument of [12, Corollary 5.4]. $\square$

Corollary 10.12. *If M is a B-supermodule that is polynomial over H, then* $\mathrm{ind}_B^G M$ *is polynomial over G. In particular, each $H^0(\lambda)$ for $\lambda \in \Lambda_p^+(T)$ is polynomial.*

Proof. It suffices to prove this for finite dimensional M, in which case $\mathrm{ind}_B^G M$ is finite dimensional too by Lemma 6.10(iii). Set

$$P := M^\tau, \quad Q := (\mathrm{ind}_B^G M)^\tau.$$

By the theorem, we just need to check that Q is polynomial over H. Since we can conjugate by W, it suffices in turn to check that Q is polynomial over the subgroup $H(1) \cong Q(1)$ embedded into G in the bottom right hand corner. Let P_0 (resp. Q_0) denote the sum of all weight spaces of P (resp. Q) with ε_n-component equal to zero, and $P_{>0}$ (resp. $Q_{>0}$) denote the sum of all weight spaces with ε_n-component greater than zero. All weights of M lie in $\Lambda(T)$, so the same is true for Q by Lemma 6.10(i). Hence,

$$P = P_0 \oplus P_{>0}, \quad Q = Q_0 \oplus Q_{>0}.$$

All $H(1)$-supermodules of weight > 0 are polynomial. Hence, $Q_{>0}$ is certainly polynomial over $H(1)$. Moreover, P_0 is polynomial over $H(1)$ by assumption.

By Lemmas 6.8 and 6.9, there is a B^+-homomorphism $\mathrm{ind}_B^G M \hookrightarrow M \otimes k[U^+]$. Considering the U^+-fixed points, we get an H-homomorphism $\mathrm{ind}_B^G M \to M$ that is injective on the B^+-socle of $\mathrm{ind}_B^G M$. Applying τ, we get an H-homomorphism $\rho : P \to Q$ whose image generates Q as a B-supermodule.

Let $B(n-1)$ denote the Borel subgroup of $Q(n-1)$ embedded into G in the top left hand corner. Then, by the previous paragraph, Q_0 is generated as a $B(n-1)$-supermodule by $\rho(P_0)$. Thus,

$$Q_0 = \sum_{1 \leq j < i < n,\ r \geq 0} \left(e_{i,j}^{(r)} \rho(P_0) + e_{i,j}^{(r)} e'_{i,j} \rho(P_0) \right).$$

Since each such $e_{i,j}^{(r)}$ and $e'_{i,j}$ centralizes $H(1)$, this shows that Q_0 is a sum of $H(1)$-homomorphic images of P_0. Since P_0 is polynomial over $H(1)$, this implies that Q_0 is polynomial over $H(1)$ too, completing the proof. $\square$

Remark 10.13. In particular, the corollary shows that the supermodules

$$V(\lambda) \cong H^0(\lambda)^\tau \tag{10.14}$$

defined in (6.12) are polynomial for $\lambda \in \Lambda_p^+(T)$. Comparing Lemma 6.13 and [2, Lemma 8.3] now shows that the $V(\lambda)$ as defined here coincide with the $V(\lambda)$ introduced in [2] for $\lambda \in \Lambda_p^+(T)$.

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